01 Jan 2022

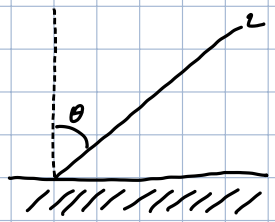
2. **Falling Rod**

A thin stick of length L with some mass distribution $\lambda(x)$ along it is initially at rest. It has one end ($x = 0$) on a horizontal table and initially makes an angle θ_0 with the vertical. Assume that the stick-table contact point has an infinite coefficient of friction (so that the end of the stick can lift off the table, but cannot slide on it). The stick is released from rest and allowed to fall to the table.

(a) Find the condition that the end of the stick initially in contact with the table does *not* rise from the table as the stick falls. Express the condition in terms of θ_0 and the mass distribution along the stick.

(b) For a stick of mass concentrated in the middle $\lambda(x) \propto (L/2)^2 - (x - L/2)^2$, what range of angles θ_0 keeps the stick in contact with the table throughout its fall?

M17M.2



$$M = \int_0^L \lambda(x)\,dx, \quad r = \int_0^L x\lambda(x)\,dx = M r_{cm}$$
$$I_{end} = \int_0^L x^2 \lambda(x)\,dx, \quad I_{cm} = \int_0^L (x - r_{cm})^2 \lambda(x)\,dx = I_{end} - M r_{cm}^2$$

a) $M\ddot{y} = -Mg \rightarrow \ddot{y} = -g$

$y = r_{cm}\cos\theta, \quad \dot{y} = -r_{cm}\dot{\theta}\sin\theta, \quad \ddot{y} = -r_{cm}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$

$\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta = \frac{g}{r_{cm}}$ ← need $\dot{\theta}$ and $\ddot{\theta}$

$\dot{\theta}$ from energy

$$E_i = Mgr_{cm}\cos(\theta_i), \quad E(\theta) = Mgr_{cm}\cos(\theta) + \frac{1}{2} I_{end}\dot{\theta}^2$$
$$E(\theta) = E_i$$
$$Mgr_{cm}\cos(\theta_i) = Mgr_{cm}\cos(\theta) + \frac{1}{2} I_{end}\dot{\theta}^2$$
$$\dot{\theta}^2 = \frac{2Mgr_{cm}}{I_{end}}\left(\cos(\theta_i) - \cos(\theta)\right)$$

$\ddot{\theta}$ from Lagrangian

$$L = T - V, \quad T = \frac{1}{2} I_{end}\dot{\theta}^2, \quad V = Mgr_{cm}\cos\theta$$
$$= \frac{1}{2} I_{end}\dot{\theta}^2 - Mgr_{cm}\cos\theta$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = I_{end}\ddot{\theta}, \quad \frac{\partial L}{\partial \theta} = Mgr_{cm}\sin\theta$$
$$\ddot{\theta} = \frac{Mgr_{cm}}{I_{end}}\sin\theta$$

$\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta = \frac{a}{r_{cm}} \leftarrow \ddot{\theta} = \frac{mgr_{cm}}{I_{end}}\sin\theta,\ \dot{\theta}^2 = \frac{2mgr_{cm}}{I_{end}}\left(\cos(\theta_i) - \cos(\theta)\right)$

$\frac{mgr_{cm}}{I_{end}}\sin^2\theta + \frac{2mgr_{cm}}{I_{end}}\left(\cos(\theta_i)\cos(\theta) - \cos^2(\theta)\right) = \frac{a}{r_{cm}}$

$\sin^2\theta + 2\left(\cos(\theta_i)\cos(\theta) - \cos^2(\theta)\right) = \frac{I_{end}}{mr_{cm}^2} \equiv R$

$1 - \cos^2\theta + 2\cos\theta_i\cos\theta - 2\cos^2\theta = R$

$(3)\cos^2\theta + (-2\cos\theta_i)\cos\theta + (R-1) = 0$

$\cos\theta = \frac{1}{6}\left[2\cos\theta_i \pm \sqrt{4\cos^2\theta_i - 12(R-1)}\right]$

$4\cos^2\theta_i - 12(R-1) > 0 \rightarrow$ allowed/still on table

$4\cos^2\theta_i - 12(R-1) < 0 \rightarrow$ disallowed/lifted off table

$\cos^2\theta_i - 3(R-1) > 0 \leftarrow R-1 = \frac{I_{end} - mr_{cm}^2}{mr_{cm}^2} = \frac{I_{cm}}{mr_{cm}^2}$

$$\boxed{\cos\theta_i > \sqrt{3\frac{I_{cm}}{mr_{cm}^2}}}$$

b) $\lambda(x) \propto \left(\frac{L}{2}\right)^2 - \left(x - \frac{L}{2}\right)^2 \leftarrow h \equiv \frac{L}{2}$, normalization h

$= \lambda_0(2xh - x^2)$

$m = \lambda_0\int_0^L (2xh - x^2)dx = \lambda_0\frac{L^3}{6}$

$f = \lambda_0\int_0^L x(2xh - x^2)dx = \lambda_0\frac{L^4}{12},\ r_{cm} = \frac{f}{m} = \frac{L}{2},\ mr_{cm}^2 = \lambda_0\frac{L^5}{24}$

$I_{end} = \lambda_0\int_0^L x^2(2xh - x^2)dx = \lambda_0\frac{L^5}{20}$

$R = \frac{I_{end}}{mr_{cm}^2} = \frac{6}{5},\ \sqrt{3(R-1)} = \sqrt{3/5}$

$\cos\theta_i > \sqrt{3/5}$

$$\boxed{\theta_i < \arccos\left(\sqrt{3/5}\right)}$$