

Shitty formulas for the preliminary exam

compiled with bitterness by Kenan Diab

December 22, 2011

Contents

1	Manifesto	2
2	Math	2
2.1	Spherical Derivatives	2
2.2	Cylindrical Derivatives	2
2.3	Sum of Geometric Series	3
3	Mechanics	3
4	Electrodynamics	4
4.1	A very important notational remark	4
4.2	Math tricks	4
4.2.1	Solving the Laplace equation	4
4.2.2	Multipole expansion	4
4.3	EM fields in matter	5
4.3.1	Boundary conditions	5
4.3.2	Work and energy	6
4.4	Potentials and fields	6
4.4.1	Warning about gauge fixing	6
4.4.2	General results	6
4.4.3	Point charge	7
4.5	Radiation	7
5	Statistical mechanics	7
5.1	Classical statistical mechanics	7
5.1.1	Canonical ensemble	7
5.1.2	Gibbs canonical ensemble	8
5.1.3	Grand canonical ensemble	8
5.2	Quantum statistical mechanics	9
5.2.1	Lattice vibrations and blackbody radiation	9
5.2.2	Quantum gases	10
6	Quantum mechanics	10
6.1	Hydrogen atom	10
6.2	Perturbation theory	11
6.2.1	Time-independent perturbations	11
6.2.1.1	Generalities	11
6.2.1.2	Ways to fuck up the hydrogen atom	11
6.2.1.2.1	Fine structure	11

6.2.1.2.2	Zeeman effect	12
6.2.1.2.3	Hyperfine structure	13
6.2.2	Time-dependent perturbations	13
6.2.2.1	Generalities	13
6.2.2.2	Fermi's golden rule	13
6.2.2.3	Stimulated emission	13
6.2.2.4	Selection rules	14
6.3	Semiclassical (WKB) approach	14
6.4	Adiabatic approximation	14
6.5	Scattering theory	15

1 Manifesto

This is not a compendium of all the formulas one might need to know for the prelims. This is a collection of **shitty formulas**. That means I'm not going to write down stuff like the Euler-Lagrange equation or Maxwell's equations or solutions to Schrodinger's equation for simple potentials. I'm not even going to describe static electromagnetic fields in vacuum or the free Fermi gas or the spectrum of the quantum harmonic oscillator. The stuff here is just the stuff that I don't use often enough to remember. Conversely, I will not discuss exotica that never seem to appear on the prelims. That means there will be no discussion of partial wave expansion, relativistic electrodynamics, Hamilton-Jacobi theory, etc.

Note that I have left some formulas un-numbered. Those are intermediate results which are probably not worth committing to memory. The numbered results are "final answers" that seem like they are probably good to know by heart, or at least good to know how to derive.

One important thing to supplement this formula list with: the formulas on the front and back covers of Griffiths's "Introduction to Electromagnetism" are all basic, necessary knowledge, and it would be wise of the reader to just staple a scan of those two pages to this thing.

2 Math

A few useful math things that you will very likely need to have memorized:

2.1 Spherical Derivatives

In spherical coordinates (r, ϕ, θ) where ϕ is the azimuthal angle and θ is the polar angle, the relevant derivatives are defined as follows:

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial f}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} \tag{1}$$

$$\nabla \times \mathbf{F} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta F_\phi) - \frac{\partial F_\theta}{\partial \phi} \right] \hat{\mathbf{r}} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r} (r F_\phi) \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r F_\theta) - \frac{\partial F_r}{\partial \phi} \right] \hat{\phi} \tag{2}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_\theta) + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi} \tag{3}$$

I won't include the Laplacian for vectors here, because if that appears on the exam, the only correct solution is to curl up in a ball and cry.

2.2 Cylindrical Derivatives

In cylindrical coordinates, the derivatives are

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \tag{4}$$

$$\nabla \times \mathbf{F} = \left[\frac{1}{r} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right] \hat{\mathbf{r}} + \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right] \hat{\phi} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r F_\phi) - \frac{\partial F_r}{\partial \phi} \right] \hat{\mathbf{z}} \quad (5)$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \quad (6)$$

2.3 Sum of Geometric Series

Also, it seems like it's generally useful to know the sum of a finite geometric series for a lot of stat mech problems. It is

$$\sum_{i=m}^{n-1} x^i = \frac{x^m - x^n}{1 - x} \quad (7)$$

3 Mechanics

There are really no formulas to be memorized for the mechanics section of the exam. However, there are a few equations, while easily derived and pretty obvious, seem to come up a lot and thus are probably worth knowing just as mental shortcuts.

The equation of motion for the central force problem can be simplified by writing the r equation of motion not in terms of r , but in terms of $u = 1/r$:

$$\frac{d^2 u}{d\theta^2} + u = -\frac{m}{L^2} \frac{d}{du} V(u) \quad (8)$$

where we remember that time and θ derivatives may be exchanged via the conserved quantity L :

$$\frac{d}{dt} = \frac{L}{mr^2} \frac{d}{d\theta} \quad (9)$$

One random fact that comes in handy when transforming into a rotating frame $\vec{\omega}$ is that for any quantity \mathbf{x} ,

$$\left(\frac{d\mathbf{x}}{dt} \right)_s = \left(\frac{d\mathbf{x}}{dt} \right)_r + \vec{\omega} \times \mathbf{x} \quad (10)$$

where s denotes the lab frame and r denotes the rotating frame. One simple consequence, for example, is the rolling without slipping condition: Let v (resp. $\vec{\omega}$) be the velocity (resp. angular velocity) of a spherically symmetric object which is rolling without slipping. Let r_{cm} be the distance from the point of contact to the center of mass. Then

$$\mathbf{v} = \vec{\omega} \times \mathbf{r} \quad (11)$$

This is also a really quick way to derive things like the Coriolis force terms, etc.

It's good to know a few moments of inertia of common objects:

$$\text{Point mass, or circle, or cylindrical shell:} \quad I = MR^2 \quad (12)$$

$$\text{Disk:} \quad I = \frac{1}{2} MR^2 \quad (13)$$

$$\text{Cylinder:} \quad I = \frac{1}{2} MR^2 \quad (14)$$

$$\text{Hollow sphere:} \quad I = \frac{2}{3} MR^2 \quad (15)$$

$$\text{Solid sphere:} \quad I = \frac{2}{5} MR^2 \quad (16)$$

$$\text{Rectangular h by w plate, through center:} \quad I = \frac{1}{12} M(w^2 + h^2) \quad (17)$$

$$\text{Rod of length L, through end:} \quad I = \frac{1}{3} ML^2 \quad (18)$$

$$\text{Rod of length L, through center:} \quad I = \frac{1}{12} ML^2 \quad (19)$$

Sometimes there are questions about incompressible fluids. The basic equations to know are the continuity equation, Bernoulli's equation, and the Navier-Stokes equation (which usually appears with some approximations that eliminate some of the terms so you can solve it). They are, in order:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \quad (20)$$

$$P + \frac{1}{2} \rho v^2 + \rho g z = \text{constant} \quad (21)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P + \mu \nabla^2 \mathbf{v} + \mathbf{f} \quad (22)$$

Here, ρ is the density of the fluid, P is pressure, v is the fluid velocity, μ is the viscosity, and \mathbf{f} are the external forces on the fluid.

4 Electrostatics

4.1 A very important notational remark

Note that since I don't know how to render the lowercase script r of Griffiths, I'm going to write it with a lowercase s throughout this document. Sorry.

$$z = |\mathbf{r} - \mathbf{r}'|$$

4.2 Math tricks

4.2.1 Solving the Laplace equation

In spherical coordinates with azimuthal symmetry - i.e. V is independent of the polar angle ϕ , the solution to the Laplace equation $\nabla^2 V = 0$ is

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta) \quad (23)$$

The $P_l(x)$ are the Legendre polynomials. Typically one only needs the first two, but I'll give the third one just in case:

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2 - 1)/2 \quad (24)$$

In cylindrical coordinates with no z -dependence, the solution to the Laplace equation is

$$a_0 + b_0 \ln r + \sum_{k=1}^{\infty} r^k (a_k \cos k\phi + b_k \sin k\phi) + r^{-k} (c_k \cos k\phi + d_k \sin k\phi) \quad (25)$$

4.2.2 Multipole expansion

One can expand the scalar potential $V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{z} \rho(r') d^3 r'$ in powers of $1/r$. The result is

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos \theta') \rho(\mathbf{r}') d^3 \mathbf{r}' \quad (26)$$

The dominant contribution is just the monopole term $V_{mon} = Q/4\pi\epsilon_0 r$. The subleading contribution is:

$$V_{dip}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}, \quad \mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d^3 \mathbf{r}' \quad (27)$$

\mathbf{p} is called the electric dipole moment.

The vector potential also has a multipole expansion. Note here that we work in the Coulomb gauge, where $\nabla \cdot \mathbf{A} = 0$. In this gauge,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{z} d^3\mathbf{r}'$$

This leads to

$$\mathbf{A} = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n(\cos \theta') \mathbf{J} d^3\mathbf{r}' \quad (28)$$

In the case where J is a current loop, one obtains that the dipole moment is the leading contribution:

$$\mathbf{A}_{dip}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}, \quad \mathbf{m} = I \int d\mathbf{a} = I\mathbf{a} \quad (29)$$

\mathbf{m} is the magnetic dipole moment of the system. It is not hard to show with some vector calculus (see Griffiths problem 5.60) how these formulas generalize to the case of a general surface or volume current. The monopole contribution is actually nonzero and related to the electric dipole moment:

$$\mathbf{A}_{mon} = \frac{\mu_0}{4\pi r} \frac{d\mathbf{p}}{dt} \quad (30)$$

The magnetic dipole moment generalizes in the obvious way:

$$\mathbf{m} = \frac{1}{2} \int (\mathbf{r} \times \mathbf{J}) d^3\mathbf{r} \quad (31)$$

4.3 EM fields in matter

4.3.1 Boundary conditions

Applying the Maxwell equations at interfaces between two materials yields the following:

$$D_1^\perp - D_2^\perp = \sigma_f \quad (32)$$

$$B_1^\perp - B_2^\perp = 0 \quad (33)$$

$$\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \quad (34)$$

$$\mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}} \quad (35)$$

For linear media, this further simplifies to:

$$\epsilon_1 E_1^\perp - \epsilon_2 E_2^\perp = \sigma_f \quad (36)$$

$$B_1^\perp - B_2^\perp = 0 \quad (37)$$

$$\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0 \quad (38)$$

$$\frac{1}{\mu_1} \mathbf{B}_1^\parallel - \frac{1}{\mu_2} \mathbf{B}_2^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}} \quad (39)$$

These are typically most useful since you can't control the bound currents or charges. But if you want to know the bound part anyway, you can use the following:

$$\rho_b = -\nabla \cdot \mathbf{P} \quad (40)$$

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \quad (41)$$

$$\mathbf{J}_b = \nabla \times \mathbf{M} \quad (42)$$

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} \quad (43)$$

All the arcana about transmission/reflection at boundaries, about wave guides, and all the other crap in chapters 8 and 9 of Griffiths can be derived just by writing down the wave equation and solving it using these boundary conditions. This is much faster and easier than memorizing random formulas about all the different cases one might encounter on the exam.

4.3.2 Work and energy

Sometimes you are asked to compute energy and momentum densities of the electric and magnetic fields in a **linear dielectric**. For **linear dielectrics only** one needs to substitute ϵ and μ for ϵ_0 and μ_0 everywhere. This modifies the usual equations as follows:

$$u_{elec} = \frac{\epsilon_0}{2} E^2 \rightarrow \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \quad (44)$$

$$u_{mag} = \frac{1}{2\mu_0} B^2 \rightarrow \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \quad (45)$$

$$\mathbf{S} = \frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \rightarrow \mathbf{E} \times \mathbf{H} \quad (46)$$

$$\varphi_{em} = \mu_0 \epsilon_0 \mathbf{S} \rightarrow \mathbf{D} \times \mathbf{B} \quad (47)$$

4.4 Potentials and fields

4.4.1 Warning about gauge fixing

In electrostatics and magnetostatics we pick $\nabla \cdot \mathbf{A} = 0$ so that a Poisson equation holds for V . However, the corresponding equation for \mathbf{A} is much more complicated. To change this asymmetrical situation, we are going to pick Lorentz gauge $\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$ so that

$$\square^2 V = -\frac{1}{\epsilon_0} \rho \quad (48)$$

$$\square^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad (49)$$

$$\square^2 \equiv \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \quad (50)$$

It is important to remember that the d'Alembertian operator \square^2 enters into the ‘‘Poisson equation’’ when you deal with nonstatic sources.

4.4.2 General results

A nonstatic source generates vector and scalar potentials according to the retarded time, since ‘‘electromagnetic news travels at the speed of light’’:

$$t_r = t - \frac{z}{c} \quad (51)$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{z} d^3\mathbf{r}' \quad (52)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_r)}{z} d^3\mathbf{r}' \quad (53)$$

The fields generated by these potentials are given by **Jefimenko's equations**:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho(\mathbf{r}', t_r)}{z^2} \hat{\mathbf{s}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{c z} \hat{\mathbf{z}} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2 z} \right] d^3\mathbf{r}' \quad (54)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{z^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c z} \right] \times \hat{\mathbf{z}} d^3\mathbf{r}' \quad (55)$$

4.4.3 Point charge

For the special case of a point charge, the retarded potentials and fields are given by the Lienard-Wiechert potentials:

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(zc - \mathbf{z} \cdot \mathbf{v})} \quad (56)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t) \quad (57)$$

Here, \mathbf{v} is the particle velocity. The expression for the fields are fucking nasty as shit because ∇s is something nontrivial. After beating yourself for a while, you get:

$$\mathbf{u} \equiv c\hat{\mathbf{z}} - \mathbf{v} \quad (58)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\hat{\mathbf{z}}}{(\hat{\mathbf{z}} \cdot \mathbf{u})^3} ((c^2 - v^2)\mathbf{u} + \hat{\mathbf{z}} \times (\mathbf{u} \times \mathbf{a})) \quad (59)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\mathbf{z}} \times \mathbf{E}(\mathbf{r}, t) \quad (60)$$

4.5 Radiation

The radiation formulas we're going to list rely on a variety of approximations, but the ones that really matter are the following: first, we're observing the radiation really far away from the source, and second, we're dropping everything except the leading terms in $1/r$.

I have never actually seen any problem that requires one to know the potentials and fields of various configurations of accelerating charges beyond the qualitative feature that the Poynting vector goes like $\sin^2 \theta / r^2$, so that power is radiated in a donut shape (i.e. no power along the direction of the acceleration). All that really matters for quantitative problem-solving is the power dissipated.

$$\text{For an oscillating electric dipole:} \quad \langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \quad (61)$$

$$\text{For an oscillating magnetic dipole:} \quad \langle P \rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3} \quad (62)$$

$$\text{For an arbitrary charge configuration:} \quad \langle P \rangle = \frac{\mu_0 \ddot{p}^2}{6\pi c} \quad (63)$$

$$\text{For a point charge:} \quad \langle P \rangle = \frac{\mu_0 q^2 a^2}{6\pi c} \quad (64)$$

$$\text{For a relativistic point charge:} \quad \langle P \rangle = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left(a^2 - \left| \frac{\mathbf{v} \times \mathbf{a}}{c} \right|^2 \right) \quad (65)$$

Yes, there has actually been a question about the relativistic correction (M05E3). The radiation reaction force is:

$$\mathbf{F}_{rad} = \frac{\mu_0 q^2}{6\pi c} \dot{\mathbf{a}} \quad (66)$$

5 Statistical mechanics

5.1 Classical statistical mechanics

5.1.1 Canonical ensemble

The idea here is that the system is in contact with a reservoir that keeps the temperature fixed. The partition function is given by

$$Z = \sum_{\{\mu\}} e^{-\beta \mathcal{H}(\mu)} \quad (67)$$

Any particular microstate μ has a probability of appearing in the ensemble at temperature T given by

$$p(\mu) = \frac{e^{-\beta\mathcal{H}}}{Z} \quad (68)$$

We have the relations

$$F = -kT \ln Z \quad (69)$$

$$\langle \mathcal{H}^n \rangle_c = (-1)^n \frac{\partial \ln Z}{\partial \beta} \quad (70)$$

F is the Helmholtz free energy. Thermodynamic quantities S, J , and μ can be calculated directly using $dF = -SdT + Jdx + \mu dN$ - e.g. $S = -\frac{\partial F}{\partial T}$. The heat capacity at constant x is the derivatives of $\langle \mathcal{H} \rangle$. The ideal gas partition function is

$$Z = \frac{1}{N!} \left(\frac{V}{\lambda^3} \right)^N \quad (71)$$

where the thermal wavelength λ is

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}} \quad (72)$$

5.1.2 Gibbs canonical ensemble

The idea here is that the system is in contact with a reservoir that keeps T and x fixed. The partition function is given by

$$\mathcal{Z} = \sum_{\{\mu\}, x} e^{-\beta\mathcal{H}(\mu) + \beta J \cdot x} \quad (73)$$

Any particular microstate (μ, x) has a probability of appearing in the ensemble at temperature T given by

$$p(\mu, x) = \frac{e^{-\beta\mathcal{H} + \beta J \cdot x}}{\mathcal{Z}} \quad (74)$$

We have the relations

$$G = -kT \ln \mathcal{Z} \quad (75)$$

$$\langle H^n \rangle_c = (-1)^n \frac{\partial \ln \mathcal{Z}}{\partial \beta} \quad (76)$$

H is the enthalpy. Thermodynamic quantities S, x , and μ can be calculated directly using $dG = -SdT - x dJ + \mu dN$. The heat capacity at constant J is the derivative of $\langle H \rangle$. The ideal gas Gibbs partition function is

$$\mathcal{Z} = \frac{1}{(\beta P)^{N+1} \lambda^{3N}} \quad (77)$$

5.1.3 Grand canonical ensemble

The idea here is that the system is in contact with a reservoir that keeps T and μ fixed. The partition function is given by

$$\mathcal{Q} = \sum_N \sum_{\{\mu\}|N} e^{-\beta\mathcal{H}(\mu) + \beta\mu N} \quad (78)$$

Any particular microstate (μ, N) has a probability of appearing in the ensemble at temperature T given by

$$p(\mu, N) = \frac{e^{-\beta\mathcal{H} + \beta\mu N}}{\mathcal{Q}} \quad (79)$$

We have the relations:

$$\mathcal{G} = -kT \ln \mathcal{Q} \quad (80)$$

$$\langle \mathcal{H}^n \rangle_c = (-1)^n \frac{\partial \ln \mathcal{Q}}{\partial \beta} \quad (81)$$

$$\langle n_i \rangle = \frac{\partial \ln \mathcal{Q}}{\partial (\beta \mu)} \quad (82)$$

Thermodynamic quantities S , J , and N can be calculated directly using $d\mathcal{G} = -SdT + Jdx - Nd\mu$. The ideal gas grand partition function is

$$\mathcal{Q} = \exp \left(e^{\beta \mu} \frac{V}{\lambda^3} \right) \quad (83)$$

5.2 Quantum statistical mechanics

5.2.1 Lattice vibrations and blackbody radiation

A lattice is usually assumed to have some discrete spectrum of $3N$ vibrational modes $\omega_\alpha(k)$ corresponding to the $3N$ degrees of freedom of the atoms in the lattice. The internal energy of the lattice is thus

$$E = E_0 + \sum_{k,\alpha} \hbar \omega_\alpha(k) \langle n_\alpha(k) \rangle$$

where $\langle n_\alpha(k) \rangle$ is determined by the canonical partition function:

$$n_\alpha(k) = \frac{\sum_n n \exp(-\beta n \hbar \omega_\alpha(k))}{\sum_n \exp(-\beta n \hbar \omega_\alpha(k))} = \frac{1}{e^{\beta \hbar \omega_\alpha(k)} - 1}$$

To proceed, you need to make some guess about the structure of the spectrum.

In the Einstein model, you assume all the ω are equal, so that

$$E = E_0 + 3N \frac{\hbar \omega \exp(-\beta \hbar \omega)}{1 - \exp(-\beta \hbar \omega)} \quad (84)$$

$$C = dE/dT = 3N k_B (T_E/T)^3 \frac{e^{-T_E/T}}{(1 - e^{-T_E/T})^2} \quad (85)$$

where $T_E = \hbar \omega / k$.

In the Debye model, you assume that $\omega(k) = vk$ (i.e. sound waves) with allowed wavevectors k given by the particle in the box so that the density of states is just $\rho = \frac{V}{(2\pi)^3}$. Then, accounting for the three polarization of acoustic vibrations, the internal energy is

$$E_0 + 3V \int_{BZ} \frac{d^3 k}{(2\pi)^3} \frac{\hbar v k}{\exp(\beta \hbar v k) - 1}$$

where the BZ means the integration is taken over the Brillouin zone. Though the shape of the BZ differs depending on the problem, the low and high temperature limits can be evaluated. Here, low and high are relative to the Debye temperature $T_D = \frac{\hbar v k_{max}}{k_B}$. At $T \gg T_D$, the integrand just becomes $k_B T$ and you recover the classical result $E = E_0 + 3N k_B T$, $C = 3N k_B$. In the low temperature regime, one can extend the integration range to all k -space, and one obtains a T^3 dependence:

$$C = k_B V \frac{2\pi^2}{5} \left(\frac{k_B T}{\hbar v} \right)^3 \quad (86)$$

Blackbody radiation is the same as the Debye model except there are only two polarizations for light and no complication about the Brillouin zone. The dispersion relation and calculations are literally the same. The energy and energy density are given by

$$\mathcal{E}(k, T) = \frac{\hbar c}{\pi^2} \frac{k^3}{e^{\beta \hbar c k} - 1} \quad (87)$$

$$\frac{E}{V} = \int dk \mathcal{E}(k) = \frac{\pi^2}{15} \left(\frac{k_B T}{\hbar c} \right)^3 k_B T \quad (88)$$

$$I = \frac{c}{4} \frac{E}{V} = \sigma T^4 = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} T^4 \quad (89)$$

The $c/4$ is the average perpendicular velocity of a gas of photons.

5.2.2 Quantum gases

Manually accounting for the symmetry constraints imposed by statistics is hard. Yet that is what you have to do to compute the partition function in the canonical ensemble. So instead we use the grand canonical ensemble. Then, dealing with statistics is easy since we know what the possible occupation numbers are for each state separately. Denoting fermions by a $\eta = -1$ and bosons by a $\eta = 1$:

$$\ln \mathcal{Q}_\eta = -\eta \sum_k \ln(1 - \eta \exp(\beta\mu - \beta\mathcal{E}(k))) \quad (90)$$

Typically, we denote $\exp(\beta\mu) \equiv z$. Then, we have the following results:

$$N_\eta = \sum_k \langle n_k \rangle = \sum_k -\frac{\partial \ln \mathcal{Q}}{\partial (\beta\mathcal{E}(k))} = \sum_k \frac{1}{z^{-1} e^{\beta\mathcal{E}(k)} - \eta} \quad (91)$$

$$E_\eta = \sum_k \langle \mathcal{E}(k) n_k \rangle = \sum_k \frac{\mathcal{E}(k)}{z^{-1} e^{\beta\mathcal{E}(k)} - \eta} \quad (92)$$

We can't proceed until we choose an energy function. A good one is the ideal gas: $\mathcal{E}(k) = \frac{\hbar k^2}{2m}$. Then $\sum_k \rightarrow V \int d^3k / (2\pi)^3$, and after a lot of crap you get

$$\beta P_\eta = \frac{g}{\lambda^3} f_{5/2}^\eta(z) \quad (93)$$

$$\beta n_\eta = \frac{N_\eta}{V} = \frac{g}{\lambda^3} f_{3/2}^\eta(z) \quad (94)$$

$$\beta \epsilon_\eta = \frac{E_\eta}{V} = \frac{g}{\lambda^3} f_{5/2}^\eta(z) \quad (95)$$

where we have defined a special function

$$f_m^\eta(z) = \frac{1}{(m-1)!} \int_0^\infty \frac{dx x^{m-1}}{z^{-1} e^x - \eta}$$

Now calculating things about the Fermi gas or Bose gas is now a matter of just screwing around with these f functions, which have a lot of random algebraic properties in various limits. But none of that is likely to be on the exam. The thing that is likely to be on the exam is the condition for Bose-Einstein condensation: $z = 1$ (or alternatively $\mu = 0$, which is when the average occupation number for the excited states turns negative, which means that any particles that are introduced in the system all have to sit in the ground state). Some key facts that are probably worth knowing is that the critical temperature T_c for condensation scales like $T^{2/3}$, and that the gas pressure becomes independent of density and scales like $T^{5/2}$. This implies the specific heat capacity of the BEC is like $T^{3/2}$.

6 Quantum mechanics

6.1 Hydrogen atom

The ground state of the hydrogen atom is

$$\psi_0 = \frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}} \quad (96)$$

where a_0 is the Bohr radius.

I might as well mention the Bohr formula: $E_n = -13.6\text{eV}/n^2$. The only other thing I'm going to write down are some of the more arcane identities involving the angular momentum operators that you may have forgotten. Recall that $L_x = yp_z - zp_y$, $L_y = zp_x - xp_z$, and $L_z = xp_y - yp_x$. We typically choose as our commuting set of observables L^2 and L_z . Then the basis of angular momentum eigenstates is indexed by two quantum numbers, l and m , and the operators $L_{\pm} = L_x \pm iL_y$ function as ladder operators. The basic relations are tabulated below:

$$L^2 f_l^m = \hbar^2 l(l+1) f_l^m \quad (97)$$

$$L_z f_l^m = \hbar m f_l^m \quad (98)$$

$$L_{\pm} f_l^m = \hbar \sqrt{(l \mp m)(l \pm m + 1)} f_l^{m \pm 1} = \hbar \sqrt{l(l+1) - m(m \pm 1)} f_l^{m \pm 1} \quad (99)$$

For orbital angular momentum L , l must be an integer and m ranges from $-l$ to l in integer steps. For spin angular momentum S , all the same relations hold, except that now half-integer values of s are acceptable.

The rules for addition of angular momenta and the derivation of the Clebsch-Gordon coefficients follow directly from these identities. If you don't know why, go read Griffiths.

6.2 Perturbation theory

6.2.1 Time-independent perturbations

6.2.1.1 Generalities Suppose $H = H_0 + \lambda H'$, where you know the eigenenergies E_n^0 and eigenstates ψ_n^0 of H_0 , and you also know they are all **non-degenerate**. Let $V_{mn} = \langle \psi_m^0 | H' | \psi_n^0 \rangle$, and let $\Delta_{mn} = E_m^0 - E_n^0$. Then, the first few corrections to the states and energies are:

$$E_n^1 = V_{nn} \quad (100)$$

$$E_n^2 = \sum_{m \neq n} \frac{|V_{nm}|^2}{\Delta_{nm}} \quad (101)$$

$$E_n^3 = \sum_l \sum_{m \neq n} \frac{V_{nl} V_{lm} V_{mn}}{\Delta_{nl} \Delta_{nm}} - V_{nn} \sum_{m \neq n} \frac{|V_{nm}|^2}{\Delta_{nm}^2} \quad (102)$$

$$\psi_n^1 = \sum_{m \neq n} \frac{V_{mn}}{\Delta_{nm}} \psi_m^0 \quad (103)$$

Don't even ask about higher order corrections to ψ_n^0 because they are ridiculous.

In the degenerate case, you need to pick a basis of the degenerate eigenstates that diagonalizes the perturbation, and then change the sums in the above formulas to be only over states that do not lie in the degenerate subspace.

6.2.1.2 Ways to fuck up the hydrogen atom It is worth noting that you probably **do not need to know the exact form of any of these corrections** - the essential part is knowing which of the unperturbed quantum numbers are broken, and what are the "good quantum numbers" that replace them in describing the states.

6.2.1.2.1 Fine structure The fine structure corrections are the ones that are order α^2 . They have two sources: first the relativistic correction, which replaces the kinetic term of the Hamiltonian with the relativistically correct kinetic term:

$$\frac{p^2}{2m} \rightarrow \sqrt{p^2 c^2 + m^2 c^4} - mc^2 = \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} + \dots$$

The first order correction breaks none of the unperturbed quantum numbers:

$$E_{rel}^1 = -\frac{E_n^2}{2mc^2} \left(\frac{4n}{l + \frac{1}{2}} - 3 \right)$$

The second fine structure correction is the spin-orbit coupling, which couples S to L as follows, mimicking the $-\mu \cdot B$ intuition from electromagnetism:

$$H_{so} = \frac{e^2}{8\pi\epsilon_0 m^2 c^2 r^3} S \cdot L$$

This breaks l as a good quantum number. Instead, we use j , which corresponds to the eigenvalues of $L + S$:

$$E_{so}^1 = \frac{E_n^2}{mc^2} \left(\frac{n(j(j+1) - l(l+1) - \frac{3}{4})}{l(l + \frac{1}{2})(l+1)} \right)$$

Adding the two corrections, we obtain:

$$E_{fs}^1 = \frac{E_n^2}{2mc^2} \left(3 - \frac{4n}{j + \frac{1}{2}} \right) \quad (104)$$

Adding in the unperturbed energy from the Bohr formula, we obtain

$$E_{nj} = -\frac{13.6\text{eV}}{n^2} \left(1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) \right) \quad (105)$$

6.2.1.2.2 Zeeman effect The Zeeman effect is a splitting of the energy levels in a magnetic field B . The perturbing Hamiltonian is:

$$H_Z = \frac{e}{2m} (L + 2S) \cdot B$$

Depending on whether B is strong or weak, it either dominates the fine structure, or it is a perturbation away from it.

In the weak field case, the Zeeman term is a perturbation, and so the good quantum numbers are n, l, j, m_j . The correction to the energy is

$$E_Z^1 = \mu_B g_J B m_j \quad (106)$$

where g_J is the Lande g-factor defined by

$$g_J = 1 + \frac{j(j+1) - l(l+1) + (3/4)}{2j(j+1)} \quad (107)$$

In the strong field case, the Zeeman term dominates, and the fine structure contributions are perturbations away from that. The good quantum numbers are n, l, m_l, m_s . The new “unperturbed energy” which replaces the Bohr formula is:

$$-\frac{13.6\text{eV}}{n^2} + \mu_B B (m_l + 2m_s)$$

The fine structure correction away from this value is determined by

$$\langle S \cdot L \rangle = \hbar^2 m_l m_s$$

Thus, the total energy in this case is

$$\frac{13.6\text{eV}}{n^3} \alpha^2 \left(\frac{3}{4n} - \left(\frac{l(l+1) - m_l m_s}{l(l + (1/2))(l+1)} \right) \right) \quad (108)$$

There is an intermediate field Zeeman effect where you actually have to do degenerate perturbation theory, diagonalizing a big subspace of hydrogen atom states, but hopefully we don't get asked about that.

6.2.1.2.3 Hyperfine structure The hyperfine structure is the order α^4 correction caused by the coupling between the electron spin and the magnetic field arising from the proton dipole moment. This breaks the degeneracy between the singlet and triplet states of the joint proton-electron wavefunction. The Hamiltonian is pretty ugly, but it is typical to only consider the perturbations of the ground state in this way, for which things simplify a lot. One finds that the hyperfine correction is:

$$E_{hf}^1 = \frac{4g_p\hbar^4}{3m_p m_e^2 c^2 a^4} \cdot \eta \quad (109)$$

where η is 1/4 for the triplet state and $-3/4$ for the singlet.

6.2.2 Time-dependent perturbations

6.2.2.1 Generalities Consider a hamiltonian $H = H_0 + H'(t)$, where the eigenstates of H_0 are known and denoted by ψ_i and the perturbation H' has **no diagonal matrix elements**. Then the solution to the Schrodinger's equation

$$\Psi(t) = \sum_i c_i(t) \psi_i \exp(-iE_i t/\hbar)$$

is given by the system of differential equations

$$\dot{c}_i = \sum_j -\frac{i}{\hbar} \langle \psi_i | H'(t) | \psi_j \rangle \exp(-i(E_j - E_i)t/\hbar) c_j \quad (110)$$

One can recursively determine the c_i starting with some initial conditions, yielding a power expansion in H' . For example, given the $t = 0$ values $c_i^{(0)}$, we have:

$$c_i^{(1)} = c_i^{(0)} + \sum_j -\frac{i}{\hbar} \int_0^t dt' \langle \psi_i | H'(t') | \psi_j \rangle \exp(-i\omega_{ij}t') c_j^{(0)} \quad (111)$$

$$c_i^{(2)} = c_i^{(0)} + \sum_j -\frac{i}{\hbar} \int_0^t dt' \langle \psi_i | H'(t') | \psi_j \rangle \exp(-i\omega_{ij}t') c_j^{(1)} \quad (112)$$

and so forth, where the resonant frequencies $\omega_{ij} = (E_j - E_i)/\hbar$.

6.2.2.2 Fermi's golden rule One can use these formulas to derive a result for the transition rate between the energy eigenstates ψ_i and ψ_f of H_0 due to $H'(t)$:

$$T_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle \psi_f | H'(t) | \psi_i \rangle|^2 \rho \quad (113)$$

where ρ is the density of states at the energy of the final state E_f .

6.2.2.3 Stimulated emission One particularly important special case is a sinusoidal perturbation, e.g. an atom coupling to an oscillating electric field. If we have a **monochromatic plane wave** $H'(t) = qEz \cos(\omega t)$, then denoting the dipole moment $p_{ij} = \langle \psi_i | qz | \psi_j \rangle$, first order perturbation theory for driving frequencies near the resonant frequency gives the transition probability

$$P_{i \rightarrow j} = \left(\frac{|p_{ij}| E_0}{\hbar} \right)^2 \frac{\sin[(\omega_{ij} - \omega)t/2]}{(\omega_{ij} - \omega)^2}$$

In terms of the energy density $u = \frac{\epsilon_0}{2} E^2$, this is

$$P_{i \rightarrow j} = \frac{2u |p_{ij}|^2 \sin[(\omega_{ij} - \omega)t/2]}{\epsilon_0 \hbar^2 (\omega_{ij} - \omega)^2} \quad (114)$$

On the other hand, for **incoherent radiation**, we need to make three modifications: first, replace the energy density u by an integration over the spectral density $\rho(\omega)$; second, replace p_{ij} by $\vec{p}_{ij} \equiv \langle \psi_i | \vec{r} | \psi_j \rangle$; third, perform a spatial average over the angular variables. If we further assume that we can approximate the spectral density by its value near the resonant frequency, the transition rate $T_{i \rightarrow j} = dP_{ij}/dt$ is given by

$$T_{i \rightarrow j} = \frac{\pi |\vec{p}_{ij}|^2 \rho(\omega_{ij})}{3\epsilon_0 \hbar^2} \quad (115)$$

6.2.2.4 Selection rules If we further specialize to the **hydrogen atom**, finding the matrix elements \vec{p} between different orbitals allows us to derive selection rules for stimulated emission - if these elements are ever zero, then a transition cannot occur. For example, at leading order (i.e. if we do not account for any broken degeneracies of the hydrogen atom), then the only allowed transitions have $\Delta m \in \{-1, 0, 1\}$ and $\Delta l = \{-1, 1\}$.

6.3 Semiclassical (WKB) approach

This approximation occurs when the potential $V(x)$ changes so slowly that you can approximately treat it as a constant and solve the Schrodinger equation that way, ignoring all corrections that contain derivatives of V . Let $p(x) = \sqrt{2m(E - V(x))}$. If $E > V(x)$ (the classical region), then

$$\psi(x) \approx \frac{C}{\sqrt{p(x)}} \exp\left(\pm \frac{i}{\hbar} \int p(x) dx\right) \quad (116)$$

If $E < V(x)$ (nonclassical region), then

$$\psi(x) \approx \frac{C}{\sqrt{|p(x)|}} \exp\left(\pm \frac{1}{\hbar} \int |p(x)| dx\right) \quad (117)$$

One can approximately calculate the tunneling probability through a very high or very wide nonclassical region that extends from $x = 0$ to $x = a$:

$$T \approx \exp\left(-2 \frac{1}{\hbar} \int_0^a |p(x)| dx\right) \quad (118)$$

Another question one might ask is how to interpolate between these approximate solutions when p is very close to zero, and thus, the WKB approximation is not valid. Here you need to do an expansion in terms of Airy functions and I have never seen a question about this on the prelims, so I am not going to discuss it.

6.4 Adiabatic approximation

Suppose we have a time-dependent hamiltonian $H(t)$ which has a nondegenerate spectrum $\{\psi_n\}$ at all times t , and is slowly varying in the sense that the matrix elements of dH/dt are all very small compared to the energy gap between the corresponding eigenstates (i.e. $|\langle \psi_m | dH/dt | \psi_n \rangle| \ll |E_m - E_n|$). Then, if a particle starts out in the state $\sum c_i(0)\psi_i(0)$, it will evolve into $\sum c_i(t)\psi_i(t)$, where

$$c_i(t) = c_i(0) e^{i\theta_i(t)} e^{i\gamma_i(t)} \quad (119)$$

$$\theta_i(t) = \frac{1}{\hbar} \int_0^t E_i(t') dt' \quad (120)$$

$$\gamma_i(t) = i \int_0^t \langle \psi_i(t') | \frac{\partial}{\partial t'} | \psi_i(t') \rangle dt' \quad (121)$$

$\theta(t)$ is called the dynamical phase - it is trivial in the sense that it generalizes the ordinary phase factors in time-independent theory to the time-dependent case. $\gamma(t)$ is the geometric phase - this is the nontrivial part that leads to fun stuff like the Aharonov-Bohm effect, etc.

The interesting examples come from the case where the hamiltonian $H(t)$ returns to the original $H(0)$ at some time t . If we assume that the time-dependence of the Hamiltonian is parameterized by some set of parameters $R_i(t)$, then we can change the integration variable from t to the R 's so that the geometric phase becomes

$$\gamma_i = i \oint \langle \psi_i | \nabla_R | \psi_i \rangle \cdot d\vec{R} \quad (122)$$

Then by Stokes's theorem, you can interpret this as being some kind of enclosed flux, with the matrix element playing the role of a vector potential.

6.5 Scattering theory

The model for scattering is that you have an incoming plane wave e^{ikz} which is scattered into a spherical wave with some angle dependence $f(\theta) \frac{e^{ikr}}{r}$ so that the total wavefunction

$$A \left\{ e^{ikz} + f(\theta, \phi) \frac{e^{ikr}}{r} \right\} \quad (123)$$

$f(\theta)$ is called the scattering amplitude. It is related to the differential cross section $D(\theta) = \frac{d\sigma}{d\theta}$ by $D(\theta) = |f(\theta)|^2$.

There are two main techniques for calculating scattering amplitudes. The first is partial wave expansion (which has never appeared on a prelim as far as I can tell, so I'm omitting it). It consists of expanding the wavefunction in a basis of spherical Hankel functions. I am only mentioning it because one can very easily prove the **optical theorem** in this formalism, which states that the total cross section is given by the forward scattering amplitude:

$$\sigma = \frac{4\pi}{k} \text{Im}(f(0, 0)) \quad (124)$$

The second method is the Born approximation. Using Green's functions, one can write the solution to the Schrodinger equation as

$$\psi(r) = \psi_0(r) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ik|r-r_0|}}{|r-r_0|} V(r_0) \psi(r_0) d^3r_0$$

Here, $\psi_0(r)$ is a solution to the free particle Schrodinger equation. For our purposes, we can take it to be a plane wave. Now, let's assume that V is localized around $r_0 = 0$ and we're only interested in the scattering at very large r , this equation simplifies to

$$\psi(r) = A e^{ikz} - \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int e^{-i\vec{k}' \cdot \vec{r}_0} V(r_0) \psi(r_0) d^3r_0$$

The Born approximations consist of recursively inserting this equation into the $\psi(r_0)$ factor in the integrand. For example, the first Born approximation keeps only one integral, yielding

$$\psi(r) = A \left(e^{ikz} - \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int e^{-i\vec{k}' \cdot \vec{r}_0 + ikz_0} V(r_0) d^3r_0 \right)$$

If we write $k' = k\hat{z}$, this equation may be rewritten

$$\psi(r) = A \left(e^{ikz} - \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}_0} V(r_0) d^3r_0 \right) \quad (125)$$

This has the form of our model from earlier, yielding:

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}_0} V(r_0) d^3r_0$$

For a spherically symmetrical potential, this simplifies considerably. Define $q = |k' - k| = 2k \sin(\theta/2)$. Then the Born approximation becomes

$$f(\theta, \phi) = -\frac{2m}{q\hbar^2} \int r_0 V(r_0) \sin(qr) d^3r_0 \quad (126)$$