Prelim Solutions: J18

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January 4, 2022

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1.1 a.

The equation relating rotating and inertial frames is:

$$\left(\frac{d\mathbf{r}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{r}}{dt}\right)_{\text{rot}} + \mathbf{\Omega} \times \mathbf{r} \tag{1}$$

where the subscript in is for the quantity in the inertial frame and rot is for it in the rotational frame. $\mathbf{\Omega}$ is angular frequency of the rotating frame relative to the inertial one. We assume that the radius vector is the same instantaneously in the rotating and inertial frame.

The accelarations are related as:

$$\begin{aligned}\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\text{in}} &= \left(\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)_{\text{rot}}\right)_{\text{in}} + \left(\frac{d}{dt}(\mathbf{\Omega}\times\mathbf{r})\right)_{\text{in}} \\ &= \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\text{rot}} + 2\,\mathbf{\Omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{\text{rot}} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})\end{aligned} \tag{2}$$

Hence,

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\text{rot}} = \frac{\mathbf{F}}{m} - 2\,\mathbf{\Omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{\text{rot}} - \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) \tag{3}$$

where the force is related to the accelaration in the inertial frame.

This leads to a system of two equations:

$$\begin{aligned}\ddot{x} &= \Omega^2 x - \frac{\alpha}{m}x + 2\Omega\dot{y} \\ \ddot{y} &= \Omega^2 y + \frac{\alpha}{m}y - 2\Omega\dot{x}\end{aligned} \tag{4}$$

1.2 b.

We are looking for periodic solutions: $x = x_0 e^{i\omega t}, y = y_0 e^{i\omega t}$. This reduces the above two equations to:

$$\begin{aligned}(\alpha/m - \Omega^2 - \omega^2)x_0 &= 2i\omega\Omega y_0 \\ (-\alpha/m - \Omega^2 - \omega^2)y_0 &= -2i\omega\Omega x_0\end{aligned} \tag{5}$$

This gives the condition (equating the ratios of x_0, y_0):

$$4\omega^2\Omega^2 = \Omega^4 + 2\omega^2\Omega^2 + \omega^4 - \alpha^2/m^2 \tag{6}$$

which is equivalent to

$$(\omega^2 - \Omega^2)^2 = \alpha^2/m^2 \implies \omega^2 = \Omega^2 \pm \alpha/m \tag{7}$$

1.3 c.

For real solutions for ω, we need $\omega^2 > 0$, i.e. $\boxed{\Omega^2 > |\alpha|/m}$.