Department of Physics, Princeton University

Graduate Preliminary Examination
Part I

Thursday, January 8, 2015
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Mechanics) and TWO out of the THREE questions in Section B (Electricity and Magnetism).

Work each problem in a separate examination booklet. Be sure to label each booklet with your name, the section name, and the problem number.

Section A. Mechanics

1. (Precession due the Sun's oblateness) In addition to the celebrated relativistic effect, the precession of the perihelion of Mercury can be affected by the deviation in the Sun's mass distribution from spherical symmetry, caused by Sun's rotation around its axis. The distortion creates a small correction to the gravitational potential, which along the plane perpendicular to the axis of rotation is:

$$\delta V(r) \;=\; -\lambda \frac{mMG}{r^n}$$

where G is the gravitational constant, M and m are the masses of the Sun and Mercury respectively, λ is a small (dimensionfull) parameter, and the power is $n > 1$.

(a) Calculate the orbital angular velocity, $\omega_{\rm orb}$, for an orbit which is close to circular, at radius r, and lies within the plane perpendicular to the axis.

(b) Write down the effective potential for radial motion and calculate the frequency of small oscillations ~~about its minimum, $\omega_{\rm osc}$~~, in terms r and the variables defined above.

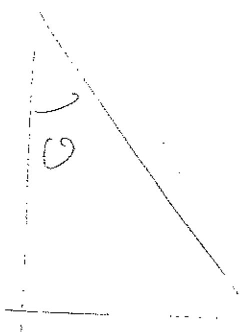
(c) From (a) and (b) find the approximate rate of precession of the perihelion (the point of closest approach) for a slightly elliptical orbit with an average radius r.

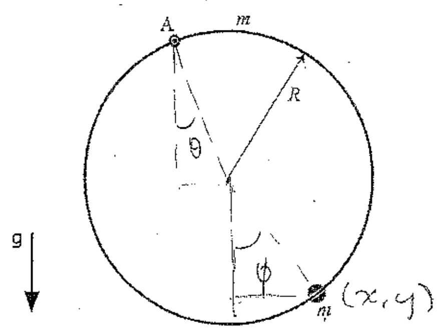
(d) Depending on the sign of λ the precession may be in the same direction as the orbital angular velocity, or opposite to it. Which is it for $\lambda > 0$?

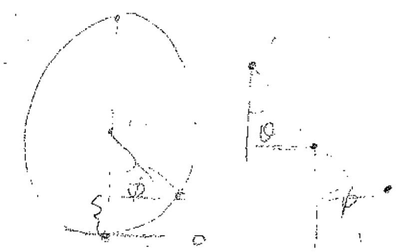
(e) What is the value of n and the sign of λ for the correction due to the Sun's mass quadrupole moment, if that is caused by oblateness (the sun's polar axis being shorter than its equatorial diameter)?

$n = 3$

2. A thin hoop of mass m and radius R is suspended from its rim (point A) and is free to rotate around point A in the plane of the hoop. A small bead of equal mass m can slide without friction on the hoop.







Find the frequencies of the normal modes for this system.

$$h_h = R + R - R\cos\theta = 2R - R\cos\theta$$

$$h_b = (R - R\cos\theta) + (R - R\cos\phi) = 2R - R(\cos\theta + \cos\phi)$$

$$T = \frac{1}{2} I \dot\theta^2 + \frac{1}{2} m(\dot x^2 + \dot y^2)$$

$$I = I_{hoop} = mR^2 + mR^2 = 2mR^2$$

$$x = R\sin\theta + R\sin\phi \qquad \dot x = R(\cos\theta\,\dot\theta + \cos\phi\,\dot\phi)$$

$$y = R(\cos\theta + \cos\phi) \qquad \dot y = -R(\sin\theta\,\dot\theta + \sin\phi\,\dot\phi)$$

$$\dot x^2 + \dot y^2 = R^2\left[(\cos\theta\,\dot\theta + \cos\phi\,\dot\phi)^2 + (\sin\theta\,\dot\theta + \sin\phi\,\dot\phi)^2\right]$$

$$= R^2(\overset{\dot\theta^2}{\cancel{\cos^2\theta\,\dot\theta^2 + \sin^2\theta\,\dot\theta^2}} + \overset{\dot\phi^2}{\cancel{\cos^2\phi\,\dot\phi^2 + \sin^2\phi\,\dot\phi^2}}$$

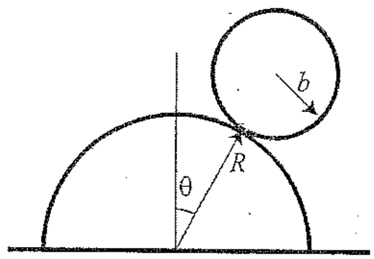
$$+ \underbrace{2\cos\theta\cos\phi\,\dot\phi\dot\theta + 2\sin\theta\sin\phi\,\dot\theta\dot\phi}_{2\cos(\theta-\phi)\dot\theta\dot\phi})$$

$$= R^2(\dot\theta^2 + \dot\phi^2 + 2\cos(\theta - \phi)\dot\theta\dot\phi)$$

$$V = mgR(2 - \cos\theta) + mgR(2 - (\cos\theta + \cos\phi))$$

$$= 2mgR(2 - \cos\theta) + mgR(2 - \cos\phi)$$

3. A uniform cylinder of mass m and radius b rolls off a fixed cylindrical surface of radius R under the influence of gravity. The axes of both cylinders are horizontal. The rolling cylinder starts from the top of the fixed cylinder with a negligibly small velocity.



(a) If we assume the cylinder rolls without slipping, find the angle θ from the vertical when it looses contact with the fixed cylinder.

(b) In practice for a finite value of μ the cylinder will start to slip before it looses contact. Find the angle when it starts to slip for $\mu = 1$.

Section B. Electricity and Magnetism

1. A small wire loop of radius a lies in the xy-plane, centered on the origin. A magnetic moment $\mathbf{m} = m\hat{\mathbf{z}}$ travels up along the z axis with constant speed v. It passes through the center of the wire loop at $t = 0$.

   (a) Compute the emf $\mathcal{E}(t)$ around the loop.
   Hint: the integral is easier if you evaluate the flux through a section of a spherical surface centered on the magnet and bounded by the wire loop rather than through the planar area bounded by the loop.

   (b) If the loop has resistance R, find the Joule heat $P(t)$ Assume the loop is fixed in position.

   (c) Now consider the case where a uniform linear charge density λ is glued to a non-conducting loop (same orientation and radius as above), and the loop is allowed to spin. What is the position of $\mathbf{m}$ at the time the loop attains its largest angular momentum, $\mathbf{L}_{max}$? What is the value of $\mathbf{L}_{max}$? Assume the dipole began its constant-velocity motion at $t = -\infty$, and that the loop was at rest then.

2. A Fresnel rhomb is an optical device used to convert linearly polarized light into circularly polarized light. As shown in Fig. 2, light hits the surface of the rhomb at normal incidence, it then undergoes two total internal reflections inside the rhomb, and then leaves the rhomb again normally.

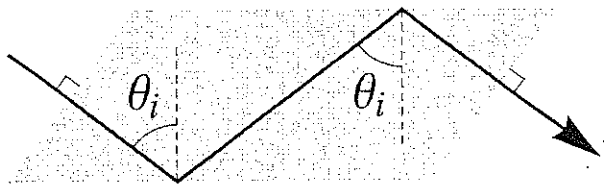


Figure 2: Fresnel rhomb.

The total internal reflections are such that each reflection generates a phase difference of 45° between the component of the light-wave that is parallel and the component that is perpendicular to the plane of incidence (the plane of the page in Fig. 2), and so after two internal reflections a lightwave that was originally linearly polarized at 45° with respect to the plane of incidence becomes circularly polarized.

(a) For a single internal reflection, find the phase shift that the reflected wave acquires relative to the incident wave assuming the electromagnetic wave is polarized in the plane of incidence.

[Hint: Let r be the ratio between the complex amplitude of the reflected wave and that of the incident wave. At total internal reflection, one has $|r| = 1$, so $r = e^{i\phi}$ for some ϕ.]

(b) Calculate the phase shift that the reflected wave acquires relative to the incident wave when the electromagnetic wave is polarized perpendicular to the plane of incidence.

(c) If each of the two total internal reflections in a Fresnel rhomb occurs at an angle of incidence of $\theta_i = 53.3°$, calculate the index of refraction n of the Fresnel rhomb relative to that of the surrounding medium.

[You might find useful the following trig. identity: $\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$. *]*

3. This problem explores some elements of a mass spectrometer. Parts (a) and (b) may be answered independently and treated non-relativistically.

   (a) An ion of charge $+q$ and mass m is accelerated through a potential V_o as shown in Fig. 3(A). It enters a region between two very long (in the direction perpendicular to the page) cylindrical electrodes of radius a and b respectively. Find the potentials $V(a)$ and $V(b)$ such that the ion moves in a circle of radius r_o.

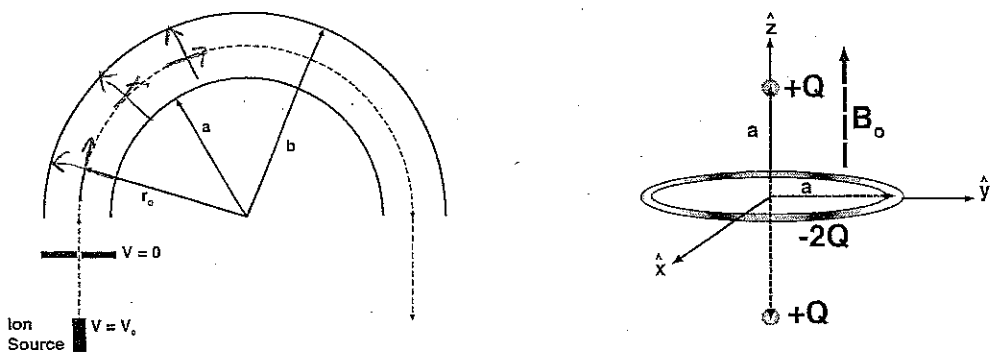


Figure 3: (A) Electrostatic filter. (B) Penning trap.

   (b) A Penning trap is used in Fourier-transform mass spectrometry. At the simplest level, the Penning trap consists of a uniform magnetic field $B_o\hat{z}$ and a quadrupole electric field. Let us assume that the electric field is generated by two positive charges $+Q > 0$ located at $\pm a\hat{z}$ and a uniformly charged ring of radius a and charge $-2Q$ centered around the origin in the xy-plane, as shown in Fig. 3(B). This setup is rotationally symmetric around the z-axis.

      i. Close to the origin, the electric field takes the form

$$\mathbf{E} = k_z z\hat{z} + k_r r\hat{r}\,,$$

      where $r = \sqrt{x^2+y^2}$. Determine k_z and k_r.

      While the general motion of an ion of mass m and charge $q > 0$ in the Penning trap is quite complicated, here we investigate only two particular cases:

      ii. Find the frequency ω_z of small oscillations around the origin in the case where the ion moves only along the z-axis.

      iii. Assume the ion moves *uniformly* along a circle of radius $R \ll a$ in the plane at $z = 0$. What is the angular frequency for this motion? Interpret the answer in the limit of large B_o.

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Graduate Preliminary Examination
Part II

Friday, May 11, 2015
9:00 am - 12:00 noon

Answer TWO out of the THREE questions in Section A (Quantum Mechanics) and TWO out of the THREE questions in Section B (Thermodynamics and Statistical Mechanics).

Work each problem in a separate booklet. Be sure to label each booklet with your name, the section name, and the problem number. Calculators are not permitted.

Section A. Quantum Mechanics

1. Consider a toy model of the Helium atom where the Coulombic interaction potential is replaced with a Hooke's law potential. If the nucleus of the atom is located at $\vec{r} = 0$ and the electrons of mass m have position vectors $\vec{r}_1$ and $\vec{r}_2$, the interaction potential is

$$V(\vec{r}_1, \vec{r}_2) = \frac{1}{2} m\omega^2 \left(\vec{r}_1^{\,2} + \vec{r}_2^{\,2}\right) - \frac{\lambda}{4} m\omega^2 \left(\vec{r}_1 - \vec{r}_2\right)^2 .$$

This model is exactly solvable. Assume $\lambda > 0$.

(a) What constraint must be imposed on λ for the system to be well-behaved? *[Hint: It may be useful to consider the center of mass and relative position vectors of the two electrons $\vec{u} = (\vec{r}_1 + \vec{r}_2)/2$ and $\vec{v} = \vec{r}_1 - \vec{r}_2$.]*

(b) What are the energy levels of this system when $\lambda = 1/2$?

(c) Taking into account the spin of the electrons, what are the degeneracies of the lowest four energy levels when $\lambda = 1/2$?

(d) Suppose the Helium atom is initially in the third excited state. It then undergoes a decay through an electric dipole transition to a lower-energy state. What are the possible energies of the emitted photon?

2. A hydrogen atom located at $\vec{r} = 0$ is initially in the $2s$ state when an ion of charge Q passes by it. Assume the ion moves with constant velocity $\vec{v} = v\hat{y}$ on a straight line whose closest approach to the hydrogen atom is $\vec{b} = b\hat{z}$, with $b \gg a_B$, where a_B is the Bohr radius. While the ion passes by, the electron in the atom experiences a time-dependent potential

$$V_1(\vec{r}, t) = \frac{Qe}{|\vec{b} + \vec{v}t - \vec{r}|}.$$

We are interested in calculating the transition probability to one of the $2p$ states. In this problem, you may assume that the $2s$ and $2p$ states are degenerate.

(a) Find an expansion of $V_1(\vec{r}, t)$ that is valid at all times and that is appropriate for calculating the transition probability in the limit $b \gg a_B$. Identify the leading term in this expansion that will give a non-vanishing transition amplitude between the $2s$ and at least one of the $2p$ states in first-order time-dependent perturbation theory.

(b) Using first-order time-dependent perturbation theory, calculate to leading order in a_B/b the probability that the atom winds up in a $2p$ state.

Some useful hydrogen atom wave-functions are:

$$\phi_{2s} = \frac{1}{2\sqrt{2\pi a_B^3}}\left(1 - \frac{r}{2a_B}\right) e^{-r/(2a_B)},$$

$$\phi_{2p,0} = \frac{z}{4\sqrt{2\pi a_B^5}} e^{-r/(2a_B)},$$

$$\phi_{2p,\pm1} = \frac{x \pm iy}{8\sqrt{\pi a_B^5}} e^{-r/(2a_B)}.$$

You may also find the following integral useful:

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{3/2}} = 2.$$

3. (a) Consider the Hamiltonian for a general time-independent, one-dimensional potential $V(x)$,

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\,.$$

Show that for an arbitrary, continuous function $\phi(x)$, the value of

$$E = \frac{\langle\phi|H|\phi\rangle}{\langle\phi|\phi\rangle}$$

gives an upper bound on the ground state energy for the potential $V(x)$.

(b) For a particle moving in a triangular potential well

$$V(x) = \begin{cases} \infty & \text{if } x < 0\,, \\ V_0 x/L & \text{if } x > 0\,, \end{cases}$$

the energy levels take the form

$$E_n = \alpha_n V_0 \left(\frac{\hbar^2}{mL^2V_0}\right)^q,$$

where α_n and q are numerical constants. Determine the value of the exponent q.

(c) Using the approach in part (a), find an estimate for the constant α_0 corresponding to the ground state in the triangular potential well. (The estimate needs not be optimal, but should be based on a reasonable variational calculation.)

Section B. Statistical Mechanics and Thermodynamics

1. **(The weather underground)** Assume the Earth is flat with its surface at $z = 0$. The solid material below the surface has a temperature-independent thermal diffusivity D. The weather above ground is a highly regular climate with sinusoidal annual (a) and daily (d) oscillations of the temperature, so the temperature at the surface as a function of time t is

$$T(t, z = 0) = T_o + T_a \cos(\omega_a t) + T_d \cos(\omega_d t) .$$

Assume the temperature at infinite depth below ground ($z \to -\infty$) is T_o.

(a) What is the temperature $T(t, z)$ below ground at time t and position $z < 0$?

(b) What is the position z closest to the surface where the annual temperature cycle is opposite to that at the surface (so that below Princeton it is instead "hottest" in January and "coldest" in July)?

(c) By what factor is the annual temperature variation attenuated at the above depth?

2. **(Atmospheric density)**

   (a) If you hold up your hand, how many molecules per second are colliding with your palm? We are interested in the order of magnitude, not the last factor of two. Approximate the atmosphere as pure oxygen (of molecular mass 32 a.m.u.).

Suppose that the atmosphere was in equilibrium at constant temperature. Then, in addition to the kinetic energy of the molecules in the air, one would have to take account of the potential energy due to gravity. Under these assumptions:

   (b) What is the probability distribution for the height h of a molecule above the surface of the earth?

   (c) How much less oxygen would you find at a height of 1 km than you do at the earth's surface?

Some relevant constants: $R \approx 8.3 \frac{\text{J}}{\text{mol K}}$, $N_A \approx 6.02 \times 10^{23}\text{mol}^{-1}$.

B+

3. **(An ideal Bose gas)**

Consider an ideal Bose gas of particles with zero spin and the dispersion relation

$$E_{\vec{k}} = c\,|\vec{k}|\,.$$

(a) Show that at high temperatures the mean energy per particle for this gas satisfies

$$U/N = \alpha T$$

with a constant α, and determine the value of that constant (in three dimensions).

Hint: a useful relation is $\int_0^\infty dx\, x^\nu\, \beta e^{-x\beta} = \nu \int_0^\infty dx\, x^{\nu-1}\, e^{-x\beta}$ (for $\beta, \nu > 0$).

(b) The pressure of such a gas with N particles becomes independent of N below a critical temperature , T_c. Explain why and calculate this temperature.

(c) Calculate the specific heat c_V of the gas for $T < T_c$.

(d) Does the transition occur if such a gas is confined to two spacial dimensions?