J12M3 Spinning Bucket

Brian Kraus

April 30, 2015

Hydrodynamics: you can start with Navier Stokes. Why not.

$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p + \nu \nabla^2 u + \rho \vec{g} \tag{1}$$

But it's an incompressible fluid and it would be mean to include viscosity so we'll set $\nabla \cdot u = 0$ and $\nu = 0$ to get the regular old Euler equations:

$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p - \rho g \vec{z} \tag{2}$$

Our bucket has been spinning forever so we're in steady state and $\partial_t = 0$. The only thing left to care about is $u \cdot \nabla u$, which looks annoying but is really just describing how a velocity line is changing as we move along it. Our spinning bucket has velocity streamlines going around in a circle, so you can imagine that

$$\begin{aligned} u \cdot \nabla u &= u_x \frac{\partial \vec{u}}{\partial x} \\ &= -\frac{u^2}{r} \end{aligned}$$

which is just centripetal acceleration towards the axis of rotation. So had we followed common sense and written a force balance equation, we could have just started at

$$\begin{aligned} -\rho \frac{u^2}{r} &= -\nabla p - \rho g \vec{z} \\ -\rho \omega^2 r &= -\frac{\partial p}{\partial r} - \rho g \vec{z} \end{aligned}$$

Let's look along a horizontal plane so that $\vec{z}$ doesn't change, and realize that the pressure is only due to atmospheric pressure, p_A, and the weight of fluid above some point:

$$p(r) = p_A + \rho g (z_0 + h(r)) \tag{3}$$

So

$$\frac{\partial p}{\partial r} = \rho g \frac{\partial h}{\partial r} \tag{4}$$

and

$$-\rho\omega^2 r = -\rho g \frac{\partial h}{\partial r} \tag{5}$$

so

$$h(r) = \frac{\omega^2 r^2}{2g} + h_{bottom} \tag{6}$$

Now you just need the initial conditions to determine h_{bottom}, the bottom of the spinning-water parabola. You know that the initial volume of water, $\pi R^2 h_0$, is constant since the water is incompressible. So just integrate our equation for $h(r)$ in space to conserve water volume:

$$\int_0^R dr 2\pi r \left(\frac{\omega^2 r^2}{2g} + h_{bottom} \right) = \pi R^2 h_0 \tag{7}$$

So solve for h_{bottom} and plug it in to get

$$h(r) = \frac{\rho\omega^2}{2g} \left(r^2 - \frac{R^2}{2} \right) + h_0 \tag{8}$$

You can solve for part b) by plugging in $h(R) = H$ and $h(0) = 0$ to find limits on ω in the above equation.

Hopefully that's on the right track, given that the integration to find h_{bottom} was the part I got wrong in the blackboard in class. Good luck!