January 2012 Problem E1

J. Alexander Jacoby[1]

[1]*Department of Physics, Princeton University, Princeton New Jersey 08544, USA*

(Dated: September 22, 2023)

This solution is pure gimmick, but it is kind of cute in my opinion.

This problem taps us on the face and says: PLEASE USE SEPARATION OF VARIABLES. We choose not to do this for fun. It is of some value to rescale these coordinates to more "natural" dimensionless choices. We choose $\tilde{y} = \frac{2\pi y}{b}$. The choice of scaling for z does not actually matter, so we choose $\tilde{z} = \frac{2\pi z}{b}$ to avoid rescaling the derivatives relative to one another (and the zero on the other side of Laplace's equation doesn't mind this either). We know that along the plane $z = a$ the solution will not depend on x and the solution must also vanish at $z = 0$. Since the solution is periodic in y, we demand a real exponential behaviour in z and sinusoidal(ish) behaviour in y. This suggests something like $e^{k\tilde{z}} - e^{-k\tilde{z}}$ (not the actual result yet). For the $\tilde{y}$ dependence, things are more complicated. I have been obsessed with the formula, $\sqrt{\pi/2}\sum_n \frac{\sin(\pi(2n+1)x)}{2n+1}$, for a very long time and this is one of the first times it has ever come in useful. It is the Fourier decomposition of a square wave. And we need the solution to turn into a squarewave at $z = a$. How convenient. Anyway, now the trick is just to stitch these solutions together. Given the

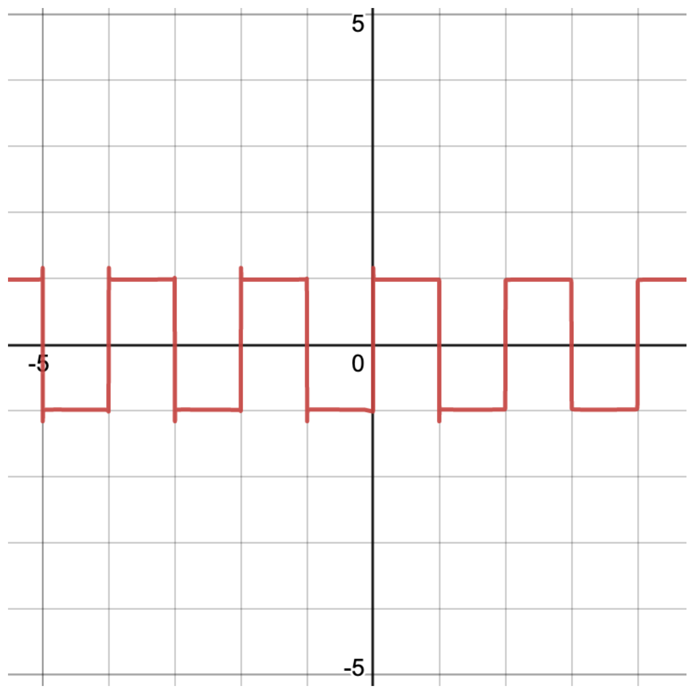

FIG. 1. Graph of $\sqrt{\pi/2}\sum_n \frac{\sin(\pi(2n+1)x)}{2n+1}$

rescaling of our variables we can see that

$$\phi(x, y, a) = \sum_n V_0 \sqrt{\pi/2}\frac{\sin((2n+1)\tilde{y})}{2n+1}\,. \tag{1}$$

So now we select the solutions $\phi = \sum_n V_0\sqrt{\pi/2}\frac{\sin((2n+1)\tilde{y})}{2n+1}\left[e^{(2n+1)\tilde{z}} - e^{-(2n+1)\tilde{z}}\right]$. But this doesn't quite nail things either. We can fix our work by setting the coefficient $(A_n)^{-1} = e^{\tilde{a}} - e^{-\tilde{a}}$. At this point, we recognize this solution as

$$\phi = V_0\sqrt{\pi/2}\sum_n \left(\frac{\sin((2n+1)\tilde{y})}{2n+1}\right)\left(\frac{\sinh[(2n+1)\tilde{z}]}{\sinh[(2n+1)\tilde{a}]}\right). \tag{2}$$

This solution is the desired one because it meets the boundary conditions (can be easily enough seen) and it has

$$\begin{aligned}
\partial_{\tilde{y}}^2 \sum_n \left(\frac{\sin\left((2n+1)\,\tilde{y}\right)}{2n+1} \right) \left(\frac{\sinh\left[(2n+1)\,\tilde{z}\right]}{\sinh\left[(2n+1)\,\tilde{a}\right]} \right) &= -\sum_n (2n+1)\sin\left((2n+1)\,\tilde{y}\right) \left(\frac{\sinh\left[(2n+1)\,\tilde{z}\right]}{\sinh\left[(2n+1)\,\tilde{a}\right]} \right) \\
\partial_{\tilde{z}}^2 \sum_n \left(\frac{\sin\left((2n+1)\,\tilde{y}\right)}{2n+1} \right) \left(\frac{\sinh\left[(2n+1)\,\tilde{z}\right]}{\sinh\left[(2n+1)\,\tilde{a}\right]} \right) &= \sum_n (2n+1)\sin\left((2n+1)\,\tilde{y}\right) \left(\frac{\sinh\left[(2n+1)\,\tilde{z}\right]}{\sinh\left[(2n+1)\,\tilde{a}\right]} \right) \qquad (3)
\end{aligned}$$

which implies it obeys Laplace's equation. Tidy.

- **Summary**

Problem includes: nothing particularly nutritional (like lettuce).