J10Q.2 - Angular Momentum

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A two particle system is in a state $|\Psi_0\rangle$ where each particle has orbital angular momentum quantum numbers $\ell = 1, m = 0$.

(a) *If the two-particle state is expanded in eigenstates of L_{tot}^2, which values of L have non-zero amplitude in the expansion? Each of these values, what is the probability that it will be found in a measurement of $|\vec{L}_{tot}^2$?*

The problem is telling us to find the product state $|1,0\rangle \otimes |1,0\rangle$ in terms of the summed angular momentum state, so we have to find the Clebsch Gordan coefficients. Since I never remember how to find these, and I found that Shankar has a good example of summing two 1/2 spin particles together. I'm going to show in detail how to get two of the Clebsch Gordan coefficients, but then it's just churning them out so it's not worth typing it out. Since we're going to be making judicious use of the raising and lowering operators, for completeness,

$$L_{\pm} = \hbar\sqrt{(l \mp m)(l \pm m + 1)}$$

The state $|\ell_1 + \ell_2, m_1 + m_2\rangle$ by definition must be $|1,1\rangle \otimes |1,1\rangle = |11,11\rangle$, where $|11,11\rangle$ is my short hand and the comma separates the two particles. To find $|2,1\rangle$, we must apply the lowering operator to L_-,

$$L_-|2,2\rangle = 2\hbar|2,1\rangle$$

Additionally, we can write the lowering operator as a sum, $L_- = L_{1,-} + L_{2,-}$ so that,

$$(L_{1,-} + L_{2,-})\,|2,2\rangle = (L_{1,-} + L_{2,-})\,|11,11\rangle = \sqrt{2}\hbar|10,11\rangle + \sqrt{2}\hbar|11,10\rangle$$

which allows us to equate $|2,1\rangle$ in terms of the summed states,

$$|2,1\rangle = \frac{1}{\sqrt{2}}\left(|10,11\rangle + |11,10\rangle\right)$$

To find the next state, we apply the lowering operator again,

$$L_-|2,1\rangle = \sqrt{6}\hbar|2,0\rangle$$

but we can write out the lowering operator as the sum, and apply it to the summed angular momentum state we found before,

$$\begin{aligned}(L_{1,-} + L_{2,-})|2,1\rangle &= \frac{1}{\sqrt{2}}(L_{1,-} + L_{2,-})\left(|10,11\rangle + |11,10\rangle\right) \\ &= \frac{1}{\sqrt{2}}\left(\sqrt{2}\hbar|1-1,10\rangle + \sqrt{2}\hbar|10,10\rangle + \sqrt{2}\hbar|10,10\rangle + \sqrt{2}\hbar|11,1-1\rangle\right) \\ &= \hbar\left(|1-1,11\rangle + 2|10,10\rangle + |11,1-1\rangle\right)\end{aligned}$$

So equating the two,

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left(|1-1,11\rangle + 2|10,10\rangle + |111-1\rangle\right)$$

We can find $|2,-1\rangle, |2,-2\rangle$ by lowering these states the same way. To find $l=1$, you have to construct it to be orthogonal to $|2,1\rangle$, and then lower it to get $|1,0\rangle, |1,-1\rangle$. Finally to construct $|0,0\rangle$, it must be orthogonal to both $|1,0\rangle$, and $|2,0\rangle$. We quote the results,

$$\begin{aligned}
|2,2\rangle &= |11,11\rangle \\
|2,1\rangle &= \frac{1}{\sqrt{2}}(|10,11\rangle + |11,10\rangle) \\
|2,0\rangle &= \frac{1}{\sqrt{6}}(|1-1,11\rangle + 2|10,10\rangle + |11,1,-1\rangle \\
|2,-1\rangle &= \frac{1}{\sqrt{2}}(|1-1,10\rangle + |10,1-1\rangle) \\
|2,-2\rangle &= |1-1,1-1\rangle
\end{aligned}$$

and,

$$\begin{aligned}
|1,1\rangle &= \frac{1}{\sqrt{2}}(|10,11\rangle - |11,10\rangle) \\
|1,0\rangle &= \frac{1}{\sqrt{2}}(|1-1,11\rangle - |11,1-1\rangle) \\
|1,-1\rangle &= \frac{1}{\sqrt{2}}(|1-1,10\rangle - |10,1-1\rangle)
\end{aligned}$$

Finally the last tricky bit,

$$|0,0\rangle = |\Psi_0\rangle = \frac{1}{\sqrt{3}}(|1-1,11\rangle - |10,10\rangle + |11,1-1\rangle)$$

We want to express $|10,10\rangle$ in terms of these states. From inspection, only $|0,0\rangle$ and $|2,0\rangle$ have these terms. If you solve for them, you'll find that,

$$\boxed{|10,10\rangle = \sqrt{\frac{2}{3}}|2,0\rangle - \sqrt{\frac{1}{3}}|0,0\rangle}$$

which allows us to read off that we have 2/3 probability we have $|\vec{L}_{\text{tot}}|^2 = 6\hbar^2$ and 1/3 probability that we have $|\vec{L}_{\text{tot}}|^2 = 0$

(b) *At time $t=0$, a coupling is switched on,*

$$H = \gamma \vec{L}_1 \cdot \vec{L}_2$$

The amplitude $f(t) = |\langle\Psi(t)|\Psi_0\rangle|^2$ oscillates, returning to the value 1 at times $t = t_n = nT$. What is the period T?

The first thing you want to do is break apart the Hamiltonian so it's uncoupled,

$$H = \frac{1}{2}\gamma(\vec{L}_{\text{tot}}^2 - \vec{L}_1 - \vec{L}_2)$$

which would let us find how the state $|\Psi_0\rangle$ evolves over time.

$$H|2,0\rangle = \frac{\gamma}{2}\left(6\hbar^2 - 2\hbar^2 - 2\hbar^2\right)|2,0\rangle = \gamma\hbar^2|2,0\rangle$$

(look at the Clebsch Jordan coefficients above if you need convincing for the L_1^2, L_2^2 eigenvalues).

$$H|0,0\rangle = \frac{\gamma}{2}\left(0\hbar^2 - 2\hbar^2 - 2\hbar^2\right)|0,0\rangle = -2\gamma\hbar^2|0,0\rangle$$

So the wavefunction evolves like,

$$|\Psi(t)\rangle = \sqrt{\frac{2}{3}} \exp\left(i\gamma\hbar t\right)|2,0\rangle - \sqrt{\frac{1}{3}} \exp\left(-2i\gamma\hbar t\right)|0,0\rangle$$

so the overlap function,

$$\begin{aligned}
\langle\Psi(t)|\Psi_0\rangle &= \frac{2}{3}\exp\left(-i\gamma\hbar t\right) + \frac{1}{3}\exp\left(2i\gamma\hbar t\right) \\
&= \left[\frac{2}{3}\cos(\gamma\hbar t) + \frac{1}{3}\cos(2\gamma\hbar t)\right] + i\left[-\frac{2}{3}\sin(\gamma\hbar t) + \frac{1}{3}\sin(2\gamma\hbar t)\right] \\
|\langle|\Psi(t)|\Psi_0\rangle|^2 &= \frac{4}{9}\cos^2(\gamma\hbar t) + \frac{1}{3}\cos^2(2\gamma\hbar t) + \frac{4}{9}\cos(\gamma\hbar t)\cos(2\gamma\hbar t) + \frac{4}{9}\sin^2(\gamma\hbar t) + \frac{1}{9}\sin^2(2\gamma\hbar t) - \frac{4}{9}\sin(\gamma\hbar t)\sin(2\gamma\hbar t) \\
&= \frac{4}{9} + \frac{1}{9} + \frac{4}{9}\left(\cos(\gamma\hbar t)\cos(2\gamma\hbar t) - \sin(\gamma\hbar t)\sin(2\gamma\hbar t)\right) \\
&= \frac{5}{9} + \frac{4}{9}\cos(3\gamma\hbar t)
\end{aligned}$$

where we see the period occurs when cosine completes a full cycle,

$$\boxed{T = \frac{2\pi}{3\gamma\hbar}}$$

When we are inbetween a cycle, cosine goes to negative 1,

$$\boxed{f\left(\frac{t_n + t_{n+1}}{2}\right) = \frac{1}{9}}$$