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Prelims Solutions
J07M.1

Ball on a Turntable

a)

We are given that the ball rolls without slipping and stays at the same point ($r_0 \neq 0$) on the turntable. Intuitively, the velocity of the ball at the contact point must be equal and opposite to that of the turntable surface at the contact point. The turntable is rotating in the counterclockwise direction, and its velocity is given by $v_{table} = r_0\Omega$, while the ball's velocity is given by $v_{ball} = -a\omega_{rot}$. Setting these equal yields the relationship $\omega_{rot} = -\frac{\Omega}{a}\Omega$.

b)

In the general case, the ball moves relative to the lab frame, with initial conditions (r_0, v_0). The ball's velocity can be broken down into two parts: one is from the motion of the turntable, given by $(\Omega\hat{z}) \times \vec{r}$, and the other is from the rotation of the ball around its center of mass, given by $\vec{\omega}_{rot} \times (a\hat{z})$. So:

$$\vec{v} = (\Omega\hat{z}) \times \vec{r} + \vec{\omega}_{rot} \times (a\hat{z}) \tag{1}$$

Note that the result from the previous section is recovered if we set $\vec{v} = 0$ and note that $\vec{\omega}_{rot}$ points along $\hat{r}$. We know that the noslip condition is provided by friction between the ball and the turntable, and this also prevents the ball from just rolling off the table; thus, the friction provides a centripetal force. Relative to the center of mass of the ball, the friction exerts a torque $\tau = (-a\hat{z}) \times \vec{F}_{friction}$. This torque is responsible for changing the direction of $\vec{\omega_{rot}}$, according to $\tau = \frac{d\vec{L}}{dt}$, where $\vec{L} = I\vec{\omega}_{rot}$ is the angular momentum. Together with the definiton of force, $\vec{F}_{friction} = m\frac{d\vec{v}}{dt}$, we have

$$I\frac{d\vec{\omega}_{rot}}{dt} = -ma\left(\hat{z} \times \frac{d\vec{v}}{dt}\right) \tag{2}$$

Differentiating (1) yields

$$\frac{d\vec{v}}{dt} = (\Omega\hat{z}) \times \vec{v} + \frac{d\vec{\omega}_{rot}}{dt} \times (a\hat{z}) \tag{3}$$

Finally, we can combine (2) and (3) to eliminate $\frac{d\vec{\omega}_{rot}}{dt}$:

$$\frac{d\vec{v}}{dt} = \Omega(\hat{z} \times \vec{v}) - \frac{ma^2}{I}\left(\left(\hat{z} \times \frac{d\vec{v}}{dt}\right) \times \hat{z}\right) \tag{4}$$

The triple cross product is easily computed to be $\frac{d\vec{v}}{dt}$, and solving for $\frac{d\vec{v}}{dt}$ in (4) yields the equation of motion for this system:

$$\frac{d\vec{v}}{dt} = \frac{\Omega}{1 + \frac{ma^2}{I}}(\hat{z} \times \vec{v}) \tag{5}$$

We identify the coefficient of the cross product as $\omega_{cm} = \frac{\Omega}{1+\frac{ma^2}{I}}$, and recognize (5) as uniform circular motion:

c)

Letting $I = \frac{2}{5}ma^2$, $\omega_{cm} = \frac{2}{7}\Omega$.