Princeton Physics Preliminary Exam Formulas

Maxwell Rosen

December 2021

1 Math

Solution to Laplace's equation in cylindrical coordinates	$V(r,\theta) = A + B\ln r + \sum_{k=1}^{\infty}\left(C_k r^k + D_k r^{-k}\right)\left(E_k \cos k\theta + F_k \sin k\theta\right)$
Solution to Laplace's equation in spherical coordinates	$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l-1}}\right)P_l(\cos\theta)$, $\Phi(\phi) = Ae^{im\phi}$
First 3 Legendre polynomials	$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \left(3x^2 - 1\right)/2$
Law of cosines	$c^2 = a^2 + b^2 - 2ab\cos\theta$

2 Mechanics

2.1 Lagrangian Mechanics

Lagrangian definition	$\mathcal{L} = T - V$
Euler-Lagrange equation	$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} = \frac{\partial\mathcal{L}}{\partial q}$
Conserved quantities of motion from a Lagrangian	If $\frac{\partial\mathcal{L}}{\partial q} = 0$, then $p = \frac{\partial\mathcal{L}}{\partial\dot{q}}$ is conserved. ($\frac{d}{dt} = 0$)
Hamiltonian in terms of Lagrangian	$\mathcal{H}(\boldsymbol{p},\boldsymbol{q},t) = \sum_{i=1}^{n} p_i\dot{q}_i - \mathcal{L}(\boldsymbol{q},\dot{\boldsymbol{q}},t)$
Hamilton's equations	$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\boldsymbol{p}}, \quad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial\boldsymbol{q}}$

2.2 Harmonic oscillators

Undamped frequency of a harmonic oscillator	$\omega_0 = \sqrt{k/m}$
Quality factor definition	$Q = \sqrt{mk}/\gamma$
Form of a damped harmonic oscillator	$\ddot{x} + \frac{\omega_0}{Q}\dot{x} + \omega_0^2 x = 0$

2.3 Central force problems

Lagrangian for a central force orbit	$\mathcal{L} = \frac{1}{2}mv_r^2 + \frac{1}{2}m(r\dot{\phi})^2 - V(r)$
Effective potential for central force problems	$V_{eff} = V(r) + \frac{\ell}{2mr^2}$. Derive from central force Lagrangian.
In a $-k/r$ potential, what do the orbits look like?	$r(\phi) = \frac{P}{1+\varepsilon\cos(\phi-\phi_0)}$ where $P = \ell/mk$, and $\varepsilon = \pm\sqrt{1 + 2E\frac{\ell^2}{mk^2}}$
Find the differential cross section	$\frac{d\sigma}{d\Omega} = \frac{b}{\sin(\theta)}\frac{db}{d\theta}$
On an ellipse, what is the periapsis?	The distance of closest approach

2.4 Rigid body motion

Torque about center of mass	$\vec{\tau} = \frac{d\vec{L}}{dt}\vert_{about\ CM}$
Rotational angular momentum	$\vec{L} = I\vec{\omega}$
Parallel axis theorem	$I = I_{CM} + MR^2$
Orbital angular momentum	$\vec{L_{CM}} = M(\vec{r}_{CM} \times \vec{v}_{CM})$
Rotational kinetic energy	$K = \frac{1}{2}I\omega^2$
Velocity of a point on a rotating body away from the center of mass	$v_{point} = v_{CM} + \omega \times r$
No slip condition	$0 = v_{cm} + \omega \times r$
Rotating reference frame derivative shift for a vector	$\left(\frac{d\mathbf{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{non-inertial}} + \Omega \times \mathbf{A}$

Using the above identity, we can derive Euler's identities. We use $\vec{A} = \vec{L}$, the spin angular momentum. In an inertial reference frame $dL/dt = 0$, so we are left with the following. I will expand it

$$I\vec{\dot{\omega}} + \vec{\omega} \times I\vec{\omega} = 0 \tag{1}$$
$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0 \tag{2}$$

Euler equations for rotating bodies	$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0$ permute ijk
Wave equation on a string in horizontal tension.	$\partial_t^2 y = v^2 \partial_x^2 y$ where $v^2 = T_H/\mu$. To remember the order of the terms in v, remember $\mu\partial_t^2 y = F$ for a small bit of string by Newton No. 2

How do you derive the fictitious forces? Use $\left(\frac{d\mathbf{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{non-inertial}} + \Omega \times \mathbf{A}$ with $\vec{v} = d\vec{x}/dt$ to write the Lagrangian for a free particle in a rotating frame. Compute the Euler Lagrange equations for the equations of motion. There are no real forces, only fictitious forces. The ABC, BAC, CAB dot cross product identities are used here.

$$\mathcal{L} = \frac{1}{2}m\left(v' + (\omega \times r')\right)^2 \tag{3}$$
$$\mathcal{L} = \frac{1}{2}m\left(v'^2 + 2v' \cdot (\omega \times r') + (\omega \times r')^2\right)^2 \tag{4}$$
$$\mathcal{L} = \frac{1}{2}m\left(v'^2 + 2r' \cdot (v' \times \omega) + r' \cdot (\omega \times r') \times \omega\right) \tag{5}$$
$$\frac{\partial \mathcal{L}}{\partial r'} = m\left((v' \times \omega) - \omega \times (\omega \times r')\right) \tag{6}$$
$$\frac{\partial \mathcal{L}}{\partial v'} = \frac{1}{2}m\left(2v' + 2(\omega \times r')\right) \tag{7}$$
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v'} = m\left(a' + (\dot{\omega} \times r') + (\omega \times v')\right) \tag{8}$$
$$ma' = -m\dot{\omega} \times r' - m\omega \times (\omega \times r') + 2mv' \times \omega \tag{9}$$

2.5 Fluids

Conservation of energy	$P + \frac{1}{2}\rho v^2 + \rho g h = \text{const.}$
Navier stokes	$\rho\frac{\partial v}{\partial_t} + \nabla \cdot (\rho\vec{v}\vec{v}) = -\nabla P + \eta\nabla^2\vec{v} + \rho F_{ext}$

3 E&M

Reflection and transmission coefficients	$t = \frac{2n_1}{n_1+n_2} = \frac{2n_1\cos\theta_i}{n_2\cos\theta_i + n_1\cos\theta_t}$ $\qquad r = \frac{n_1-n_2}{n_1+n_2}$
Maxwell Stress	$T_{ij} = \varepsilon[E_iE_j + c^2B_iB_j - \frac{1}{2}(E^2 - c^2B^2)\delta_{ij}]$

Capacitance definition and energy	$Q = CV \quad E = \frac{1}{2}QV = \frac{1}{2}CV^2$
Capacitance of parallel plates	$C = \frac{\varepsilon_0 A}{d}$
B field from a wire	$B = \frac{\mu_0 I}{2\pi r}$
B field inside a solenoid	$B = \mu N I$
D in terms of E	$D = \varepsilon E = \varepsilon_0 E + P$
H in terms of B	$H = B/\mu = B/\mu_0 - M$
Discontinuity condition for electric field	$\Delta E_{\perp} = \sigma_B/\varepsilon \quad \Delta E_{\parallel} = 0$
Discontinuity condition for magnetic field	$\Delta B_{\parallel} = \mu \vec{k} \quad , \quad \Delta B_{\perp} = 0$

Perpendicular means perpendicular to the plane of the surface. Both of these conditions are for the displacement fields D, H instead of for E, B really.

Superconducting boundary condition	$B = 0$ on the surface

3.1 Method of images conditions

Method of images: charge condition	Opposite charge and position
Method of images: dipole charge on conducting boundary	Flip along the axis perpendicular to the plane
Method of images: magnetic dipole condition	Mirror the dipole across the boundary
Point charge and sphere	$q' = -qR/a$ and $b = R^2/a$
Electric and magnetic potential definitions	$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \quad , \quad \vec{B} = \nabla \times A$
Poynting vector and E&B field momentum	$\vec{S} = 1/\mu E \times B = E \times H \quad , \quad \vec{g} = \varepsilon E \times B = D \times B$
Energy density in field	$U/V = \frac{1}{2}\left[\varepsilon E^2 + \frac{1}{\mu}B^2\right]$
Brute force magnetic potential (A) integral	$A = \frac{\mu_0}{4\pi}\int \frac{J(\vec{r})}{\lvert r \rvert} d^3 r$
Brute force electric potential (V) integral	$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')}{\lvert r - r' \rvert} d^3 r'$
Brute force magnetic field (B) integral	$B = \frac{\mu_0}{4\pi}\int \frac{d\vec{J} \times \hat{r}}{\lvert r \rvert^2}$
Brute force electric field (E) integral	$E = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r})}{\lvert r \rvert^2} \hat{r} dr$
Magnetic moment integral	$\mathbf{m} = \frac{1}{2}\iiint_V \mathbf{r} \times \mathbf{j}\, \mathrm{d}V$
Torque on a dipole	$\vec{\tau} = \vec{m} \times \vec{B_{ext}}$
Energy of a dipole in a magnetic field	$U = -m \cdot B_{ext}$. Remember minus because dipoles want to align with B field
Force on a dipole	$F = \nabla(m \cdot B_{ext})$

Angular momentum of E&B field	$\vec{l} = \vec{r} \times \vec{g} = \vec{r} \times \varepsilon(E \times B)$
Magnetic field from dipole	$\vec{B}_{dip} = \frac{\mu_0}{4\pi r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$
Magnetic potential from a dipole	$\vec{A} = \frac{\mu_0 \vec{m}\times\hat{r}}{4\pi r^2}$
Power and force of an accelerating charge	$P = \frac{\mu_0}{6\pi c} q^2 a^2 \quad , \quad F = \frac{\mu_0}{6\pi c} q^2 \dot{a}$
Current in terms of electric field	$J = \sigma E$
Polarization in terms of displacement	$P = Nq\vec{r}$
TE vs TM vs TEM	TE: $E_z = 0$, TM: $B_z = 0$, TEM: $B_z = E_z = 0$

Let's derive a dispersion relation in a conductive medium.

$$\nabla \times B = \mu J + \frac{1}{c^2}\frac{\partial E}{\partial t} \tag{10}$$

$$\nabla \times \nabla \times B = \sigma\mu\nabla \times E + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla \times E) \tag{11}$$

$$-\nabla^2 B = -\sigma\mu\frac{\partial B}{\partial t} - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} \tag{12}$$

$$k^2 B = i\omega\sigma\mu B + \frac{1}{c^2}\omega^2 B \tag{13}$$

$$c^2 k^2 = i\omega\mu\sigma c^2 + \omega^2 \tag{14}$$

4 Quantum

4.1 Mathy things

Unitary definition	$f(-a) = f^\dagger(a)$ i.e. inverse = conjugate transpose
Commutator $[A, B]$	$= AB - BA$
Expectation value of an operator	$< A >=< \psi \vert A \vert \psi >= \sum c_n^2 A_n$
Orthonormality and eigenvalues	$< \psi_n \vert \psi_m >= \delta_{nm} \quad , \quad \hat{A}\psi_n = a_n\psi_n \quad , \quad \psi = \sum_n c_n\psi_n \quad , \quad c_n =< \psi_n \vert \psi > \quad , \quad P_n = \vert c_n \vert^2$
Hermitian operator definition and important property	$H = H^\dagger \quad , \quad < v \vert Hw >=< H^\dagger v \vert w >$
Inner product (vectors and functions	$< v \vert w >= \sum v_i^* w_i = v^\dagger w \quad , \quad < f \vert g >= \int f(x)^* g(x) dx$
Energy and momentum in QM	$E = hf = \hbar\omega \quad , \quad p = \hbar k = h/\lambda$
Ehrenfest Theorem	$\frac{d<A>_t}{dt} =< \frac{\partial A}{\partial t} > + \frac{i}{\hbar} < [H, A] >$
Generalized uncertainty principle	$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} < [A, B] >\right)^2$
Density operator and matrix elements	$\hat{\rho} = \sum p_k \vert \psi_k >< \psi_k \vert \quad , \quad A_{ij} =< e_i \vert \hat{A} \vert e_j >$

4.1.1 Fundamentals related to 1D SE

Energy operator $\hat{H}$	$\hat{H} = \frac{p^2}{2m} + V(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)$
Schrodinger equation	$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$
Position and momentum operators	$\hat{x} = x \quad , \quad \hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$
Time Independent Schrodinger equation	$\hat{H}\psi = E\psi$

4.2 1D problems

Infinite square well	$\sqrt{\frac{2}{L}}\sin\left(n\pi x/L\right)$
Infinite square well Energy levels	$E_n = \frac{\hbar^2 n^2 \pi^2}{2mL}$
Delta function discontinuity condition. V(x) = a $\delta(x)$	Integrate SE over each side of the δ function and show $\Delta\partial_x\psi(x)\|_{x_0} = \frac{2ma}{\hbar^2}\psi(x_0)$
1D Time independent wave function construction. Determine wavefunction from initial condition	$\psi_n(x,t) = \psi_n(x)e^{-iE_n t/\hbar}$, $\psi(x,t) = \sum_n c_n\psi_n(x,t)$, $H\psi_n = E_n\psi_n$, $c_n = \langle\psi_n\|\psi(x,0)\rangle$

4.2.1 Harmonic oscillator

Harmonic oscillator raising and lowering operators and normalization. $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$	$X = x\sqrt{\frac{m\omega}{\hbar}}$, $P = p\frac{1}{\sqrt{m\hbar\omega}}$ $a_- = \frac{X+iP}{\sqrt{2}}$, $a_+ = \frac{X-iP}{\sqrt{2}}$
Determine ground state of harmonic oscillator	$a_-\psi_0 = 0 = \left(\frac{X+iP}{\sqrt{2}}\right)\psi_0 = 0$
Energies of 3D SHO	$E_n = \hbar\omega(f/2 + n)$
Raising and lowering operations on wavefunctions	$a_-\psi_n = \sqrt{n}\psi_{n-1}$, $a_+\psi_n = \sqrt{n+1}\psi_{n+1}$
Ground state of SHO	$\psi_0 = N_0 e^{-x^2 m\omega/2\hbar}$

4.3 2 particle systems

Wavefunction and energies of 2 non-interacting distinguishable particles	$\psi(r_1,r_2) = \psi_a(r_1)\psi_b(r_2)$, $E = E_a + E_b$
Wavefunction and energies of 2 indistinguishable identical particles	$\psi_\pm(r_1,r_2) = A[\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)]$, $+bosons - fermions$, $E = E_a + E_b$

4.4 3D problems and hydrogen atom

Bohr condition	$l = \hbar n$ angular momentum is quantized
Spherical harmonic $Y_{0,0}$	$Y_{0,0} = 1/\sqrt{4\pi}$
Spherical harmonic $Y_{1,0}$	$Y_{0,0} = N_{10}\cos\theta$
Spherical harmonic $Y_{1,\pm1}$	$Y_{0,0} = N_{11}\sin\theta e^{i\phi}$
Ground state wavefunction for hydrogen	$\mathrm{e}^{-r/a_B}/\sqrt{\pi a_B^3}$
Bohr Radius	$a_B = \frac{4\pi\varepsilon_0}{e^2}\frac{n^2\hbar^2}{m_e}$ where n=1
Hydrogen energy levels	$E = \frac{1}{2}U = \frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2$. Use Virial theorem.

4.5 Orbital stuff, angular momentum, and spin

Orbital angular momentum EV equations	$L^2\|\ell m >= \hbar^2\ell(\ell+1)\|\ell m >$ and $L_z\|\ell m >= \hbar m\|\ell m >$
Commutators of angular momentum operators	$[L_x, L_y] = i\hbar L_z$ and permutations therefore
Raising and lowering angular momentum operators	$L_\pm = L_x \pm iL_y$
Eigenvalues equation of raising and lowering operators	$L_\pm\|\ell m >= \hbar\sqrt{\ell(\ell+1) - m(m\pm 1)}\|\ell(m\pm 1) >$. Figure out signs of signs by using $L_+\|\ell,\ell >= 0$ and $L_-\|\ell,-\ell >= 0$
How do the spin and orbital angular momentum EV equations differ	They don't. Just use $S = L$ and, $s = l$
Pauli spin matrix	$\begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix}$. 1st column positive, 2nd column negative.
What are Pauli spin matrices used for?	Measuring the wavefunction χ in the spin directions. $c_{n,z} = \sigma_z\chi$
Eigenvalues of E for $H = \sigma_1 \cdot \sigma_2$	Singlet $E = \frac{\hbar^2}{4}$. Triplet $E = -\frac{3\hbar^2}{4}$

4.6 Perturbation theory

Let's derive the 1st order corrections. First, expand EVERYTHING to an order in λ

$$\left(\hat{H}^0 + \lambda\hat{H}^1\right)\left(\psi_n^0 + \lambda\psi_n^1 + \ldots\right) = \left(E_n^0 + \lambda E_n^1 + \ldots\right)\left(\psi_n^0 + \lambda\psi_n^1 + \ldots\right) \tag{15}$$

Now group in like powers of λ. The first order terms in λ read

$$H^0\psi + n^1 + H^1\psi_n^0 = E_n^0\psi_n^1 + E_n^1\psi_n^0 \tag{16}$$

Now comes the tricky part. We want to inner product everything with $< \psi_n^0|$ to get rid of some inner products.

$$\left\langle\psi_n^0|H^0\psi^1\right\rangle + \left\langle\psi_n^0|H^1\psi_n^0\right\rangle = \left\langle\psi_n^0|E_n^0\psi_n^1\right\rangle + \left\langle\psi_n^0|E_n^1\psi_n^0\right\rangle \tag{17}$$
$$E_n^0\left\langle\psi_n^0|\psi^1\right\rangle + \left\langle\psi_n^0|H^1\psi_n^0\right\rangle = E_n^0\left\langle\psi_n^0|\psi_n^1\right\rangle + E_n^1\left\langle\psi_n^0|\psi_n^0\right\rangle \tag{18}$$
$$\left\langle\psi_n^0|H^1\psi_n^0\right\rangle = E_n^1 \tag{19}$$
$$\tag{20}$$

To get the wavefunctions, first we start by grouping the first order terms in groups of ψ

$$\left(H^0 - E_n^0\right)\psi_n^1 = -\left(H^1 - E_n^1\right)\psi_n^0 \tag{21}$$

We then expand ψ_1 as a linear combination of the eigenstates of the unperturbed Hamiltonian to deal with the $H^0\psi_n^1$ term.

$$\sum\left(E_m^0 - E_n^0\right)c_m\psi_m^0 = -\left(H^1 - E_n^1\right)\psi_n^0 \tag{22}$$

Then we take the inner product with the same eigenstate ψ_m.

$$\sum\left(E_m^0 - E_n^0\right)c_m < \psi_m^0|\psi_m^0 >= - < \psi_m^0|H^1\psi_n^0 > + E_n^1 < \psi_m^0|\psi_n^0 > \tag{23}$$

This allows us to simplify some of the inner products. We get rid of the $< \psi_n|\psi_m >$ inner product by saying $n \neq m$, which gives 0 on the RHS anyway.

$$\left(E_m^0 - E_n^0\right)c_m = - < \psi_m^0|H^1\psi_n^0 > \tag{24}$$
$$c_m = \frac{< \psi_m^0|H^1\psi_n^0 >}{(E_n^0 - E_m^0)} \tag{25}$$
$$\psi_n^1 = \sum_{m\neq n}\frac{< \psi_m^0|H^1\psi_n^0 >}{(E_n^0 - E_m^0)}\psi_m^0 \tag{26}$$

1st order PT Energies	$E_n^1 =< \psi_n^0\|H_1\|\psi_n^0 >$
1st order PT wavefunctions	$\psi_n^1 = \sum_{m\neq n}\psi_m^0\frac{<\psi_m^0\|H_1\|\psi_n^0>}{E_n^0 - E_m^0}$

2nd order PT Energies	$E_n^2 = -\sum_{m\neq n} \frac{\lvert (H_1)_{m,n}\rvert^2}{E_m^0 - E_n^0}$
Coefficient for the time dependent transition probability	$m_{i\to f} = -\frac{i}{\hbar}\int_0^t dt' e^{i(E_f - E_i)t'/\hbar} < f\lvert V(t')\rvert i >$
Fermi's golden rule	$P_{i\to f} \approx \frac{2\pi t}{\hbar}\lvert\langle f\lvert F\rvert i\rangle\rvert^2 \delta\left(E_f - E_i - \hbar\omega\right) + \frac{2\pi t}{\hbar}\left\lvert\left\langle f\left\lvert F^{\dagger}\right\rvert i\right\rangle\right\rvert^2 \delta\left(E_f - E_i + \hbar\omega\right)$
Fermi's golden rule transition rate	$\Gamma = \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{2\pi}{\hbar}\lvert\langle F\rangle\rvert^2\delta(\Delta E - \hbar\omega) + \frac{2\pi}{\hbar}\left\lvert\left\langle F^{\dagger}\right\rangle\right\rvert^2\delta(\Delta E + \hbar\omega)$
Fermi's golden rule density of states	$\Gamma_{\text{tot}} = \frac{2\pi}{\hbar}\rho\left(E_i + \hbar\omega\right)\lvert\langle f\lvert F\rvert i\rangle\rvert^2 + \frac{2\pi}{\hbar}\rho\left(E_i - \hbar\omega\right)\left\lvert\left\langle f\left\lvert F^{\dagger}\right\rvert i\right\rangle\right\rvert^2$
Fermi's golden rule easy. Density of states	$R = \frac{dP}{dt} = \frac{2\pi}{\hbar}\lvert < f\lvert V\rvert i > \rvert^2\rho(E)$

4.7 Variational Method

Statement about the ground energy	$E_0 \leq \frac{<\psi_\sigma\lvert H\rvert\psi_\sigma>}{<\psi_\sigma\lvert\psi_\sigma>}$

4.8 EM in QM

Hamiltonian with magnetic and electric correction	$H = \frac{(p - qA)^2}{2m} + q\phi$

4.9 WKB

How to approximate ψ far from the turning points.	$p(x) = \sqrt{2m\left(E - V(x)\right)} \to \psi(x) \approx \frac{A}{\sqrt{p(x)}}\exp\left(\pm\frac{i}{\hbar}\int_0^x dy p(y)\right)$

4.10 Scattering

1D scattering	Set waves on either side for scattering and reflecting, then match BC in continuity in value and slope.
First Born aproximation	$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int d^3\vec{r'} e^{i(k_i - k_f)\cdot\vec{r'}} V(\vec{r'})$ then $\frac{d\sigma}{d\omega} = \lvert f\rvert^2$
First Born approximation spherical symmetry	Take $\kappa = 2k\sin(\theta/2)$ then $f(\theta,\phi) = -\frac{2m}{\hbar^2\kappa}\int_0^\infty dr' r' V(r')\sin(\kappa r')$

5 Statistical Mechanics

Probability in terms of multiplicity	$P(n) = 1/\Omega(E)$
Entropy definition	$S(E) = k\ln\Omega$
Multiplicity of 2 systems together	$\Omega_{12} = \Omega_1\Omega_2$
Entropy of two systems	$S_{12} = S_1 + S_2$
dU	$dU = TdS - PdV + \mu N$
Free energy	$F = U - TS$
Free energy from partition function	$F = -kT\ln Z$
Gibbs Free energy	$G = U - TS + PV$

Heat capacity definition	$c = \frac{\partial E}{\partial T}$
2nd law of thermodynamics (kind of)	$\Delta S = \int_{T_1}^{T_2} \frac{c(T)}{T} dt$
Sterling's approximation	$\ln(N!) \approx N \ln N - N$
Multiplicity in terms of Boltzmann factors	$\Omega = 1/Z \sum e^{-\beta E_n}$
$< E >$ in terms of Z	$< E >= U = -\frac{\partial}{\partial \beta} \ln Z$
σ_E^2	$\sigma_E^2 = \frac{\partial^2}{\partial \beta^2} \ln Z$
Z_{12}	$Z_{12} = Z_1 Z_2$
Equipartition theorem	Each particle has energy distributed evenly on average across all degrees of freedom. Total energy is $< U >= \frac{f}{2} kT$ per particle.
Density of states of a Fermi gas	$g(E) = \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E}$

What is the preferred way to derive the density of states? State many expressions. $\int g(\varepsilon) d\varepsilon = \int d\vec{n} = \int d\frac{\vec{kL}}{2\pi}$. The 2π comes from imposing periodic boundary conditions. It would be π for Dirichlet BC. In 3D, take this in polar coordinates in k space and this becomes. $= \left(\frac{L}{2\pi}\right)^3 \int 4\pi k^2 dk$. For a dispersion relation, we can use $E = \frac{\hbar^2 k^2}{2m}$ so $dE = \frac{\hbar^2 k}{m} dk$. Plugging this back in, we get the above expression for a density of states.

The number of particles covers up to the fermi-level in 1 octant of a sphere with 2 fold degeneracy from spins. $2 \cdot \frac{1}{8} \frac{4}{3} \pi n_f^3 = N$. Then using energies $E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} (\frac{\pi n}{L})^2$. The wavenumber here comes from the infinite potential box condition. Then you can find that $\varepsilon_F = \frac{\hbar^2}{2m} \left(\frac{3N\pi^2}{V}\right)^{2/3}$. This is also the chemical potential at 0 temperature.

Fermi-Dirac distribution	$f_{FD} = \frac{1}{e^{\beta(\varepsilon - \mu)} + 1}$
Bose-Einstein distribution	$f_{BE} = \frac{1}{e^{\beta(\varepsilon - \mu)} - 1}$
γ for degrees of freedom	$\gamma = (f+2)/f$
Partition function of an ideal gas	$Z = (V/\lambda^3)^N / N!$ where $\lambda = \frac{h}{\sqrt{2\pi m k T}}$
What is constant over an adiabatic curve?	$PV^\gamma = const$
What does adiabatic mean	$dQ = 0$
What does reversible mean?	$dS = 0$
What does isothermal mean	$dU = 0$
Clausius Clapyron equation. State the main way to get there	$\frac{dP}{dT} = \frac{L}{T(V_{gas} - V_{liquid})}$. Say the Gibbs energies are the same $G_{gas} = G_{liquid}$ then solve for dP/dT
Definition of latent heat	$L = T(S_{gas} - S_{liquid})$
Carnot engine efficiency	$\eta = 1 - Q_C/Q_H = 1 - T_C/T_H$
Carnot fridge efficiency	c.o.p. $= 1 - Q_H/Q_C = 1 - T_H/T_C$

Maxwell's relations	Look at the 2nd derivatives of the thermodynamic identities and relate the two ways of taking it in order
How does a phase transition relate to free energy	Gibbs free energy doesn't change
How does a phase transition relate to chemical potential	Chemical potential is the same at a phase transition
Van der waals equation condition for the critical P,V and T	$\partial_V P\|_{T_C,V_C} = 0$ and $\partial_V^2 P\|_{T_C,V_C} = 0$. 3rd equation is the equation of state

5.1 Polymers / Random Walks

What is the mean displacement of a random walker	$< x >= 0$ if it's a uniform distribution
What is the standard deviation of a random walker?	$< x^2 >= \sigma^2 = Na^2$ where a is the step size
What is the probability density function (Standard normal distribution	$P(x) = \frac{1}{\sqrt{2\pi\sigma}} exp(-x^2/2\sigma^2)$
Freely rotating chain variance	$< x^2 >= Na^2(\frac{1+\cos\theta}{1-\cos\theta})$
Work done on a polymer	$dW = F \cdot dr.$ vs ideal gas $dW = -pdV$

5.2 Ising Model

Mean field approximation	Each particle sees a spin of the average spin of the total system. Energy of 1 spin particle is $J < \sigma > \sigma_i$
Partition function for 1D Ising model. $H = -J\sum_{i,j}\sigma_i\sigma_{i+j} - B\sum_i \sigma_i$	$(2^N \cosh(\beta(M+B)))^N$ where $M = J < \sigma >$
Average spin definition	$< \sigma >= M/N$
Magnetization in terms of distribution function	$M = Nm = \frac{1}{\beta}\frac{\partial \ln Z}{\partial H}$
Critical Temperature condition	When the slope of $< \sigma >= 1$

5.3 Brownian motion