M99T.3 SURFACE WAVES

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1. Problem

Consider waves on a liquid surface where the restoring force is produced by surface tension. Assume there is a single polarization and the dispersion relation is:

$$\omega^2 = \frac{\gamma}{\rho} k^3,$$

where γ is the surface tension of the liquid, ρ is its density, ω is the frequency of the waves, and k is the wavenumber of the waves. Our goal is to find the contribution of these waves to the low temperature heat capacity of the liquid.

At low temperature T what are the total energy and heat capacity, per unit volume, of these surface waves? Your answer may involve a constant defined by a dimensionless integral. You need not compute its value (denote it I). However, you should explain why, and under what conditions, it is OK to set the upper limit to ∞.

2. Total Energy

The energy of the waves is given by:

$$E = \int \epsilon\, g(\epsilon) n_b(\epsilon)\, d\epsilon = \int \epsilon(k)\, g(k) n_b(\epsilon(k))\, dk.$$

I like to work in momentum space. The energy per state $\epsilon(k)$ is given by:

$$\epsilon(k) = \hbar\, \omega(k) = \hbar \sqrt{\frac{\gamma}{\rho}} k^{3/2}.$$

In 2D space, the density of states $g(k)$ is[1]:

$$g(k) = \frac{A}{(2\pi)^2} 2\pi k.$$

And of course, $n_b(\epsilon(k))$ is the Bose-Einstein distribution. For small T, the exponential term is very large compared to the one in the denominator, so we can approximate the distribution thusly:

$$n_b(\epsilon) = \frac{1}{e^{\epsilon/k_B T} - 1} \approx e^{-\epsilon/k_B T}.$$

Then energy is:

$$E = \frac{A\hbar}{2\pi} \sqrt{\frac{\gamma}{\rho}} \int_0^\infty k^{5/2}\, e^{-\hbar\sqrt{\frac{\gamma}{\rho}} k^{3/2}/k_B T}\, dk. \tag{1}$$

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[1]Note: if multiple polarizations were allowed, we would simply multiply $g(k)$ by the number of polarizations, since each polarization allows another set of identical states to occupy. This is akin to allowing both spins for electrons.

To turn the integral into a unitless integral, we make the substitution:

$$u = \hbar \frac{1}{k_B T} \sqrt{\frac{\gamma}{\rho}} k^{3/2},$$

$$du = \hbar \frac{3}{2 k_B T} \sqrt{\frac{\gamma}{\rho}} k^{1/2}.$$

This replacement gives energy:

$$E = \frac{A}{3\pi} \left(\frac{1}{\hbar^2} \frac{\rho}{\gamma} \right)^{2/3} (k_B T)^{7/3} \int_0^\infty u^{4/3}\, e^{-u}\, du.$$

We are told the constant, dimensionless integral is denoted by I, so:

$$E = \frac{A}{3\pi} \left(\frac{1}{\hbar^2} \frac{\rho}{\gamma} \right)^{2/3} (k_B T)^{7/3} I.$$

3. Heat Capacity

The heat capacity is the derivative of the energy with respect to temperature. We cannot, however, simply take the derivative of the energy we found since the dimensionless integral depends on T.

We instead take the temperature derivative inside equation (1) to obtain:

$$C_V = \frac{A \hbar^2 \gamma}{2\pi k_B T^2 \rho} \int_0^\infty k^4\, e^{-\hbar \sqrt{\frac{\gamma}{\rho}} k^{3/2} / k_B T}\, dk.$$

If we make the same substitution, we get:

$$C_V = \frac{A k_B^{7/3}}{3\pi} \left(\frac{T}{\hbar} \right)^{4/3} \left(\frac{\rho}{\gamma} \right)^{2/3} \int_0^\infty u^{7/2}\, e^{-u}\, du.$$

We have reduced the problem to terms of a dimensionless integral as requested.

We choose to let the integral run to ∞ to make the integral a known value of the Gamma function (the dimensionless integral above is $Gamma[9/2]$). This is appropriate to do since the Bose-Einstein statistics go to zero quickly for larger values of k, so these error terms contribute negligibly to the integral (the occupation statistics goes like a decaying exponential, shown above). Thus, we can set the upper limit to ∞ when the temperature is low enough that our exponential approximation for the Bose-Einstein statistics is valid.