M99T.1 - Otto Cycle (Solution by Jim Wu)
Calculate the efficiency of a car engine modeled by the Otto cycle depicted in the diagram. The cycle consists of two isochorous and two adiabatic processes between volumes V_1 and V_2.
The working medium's equation of state is given by

$$p = \frac{n}{V}\left(RT + \frac{n\alpha RT}{V}\right) \qquad , \qquad -\frac{V_1}{2} < n\alpha < V_1$$

(the ideal gas law corrected by the first virial coefficient $RT\alpha$), and the medium's molar heat capacity C_V (at constant volume) remains approximately unchanged throughout the cycle. Find the efficiency of the depicted cycle in terms of V_1, V_2, R, C_V, and $n\alpha$.

Solution:

Since we have to isochoric processes, then we only need to consider the work during the adiabatic processes. Since there is no heat exchange during this process, then the work done by the engine on the environment corresponds to the negative change in internal energy during that process:

$$\Delta W_{adiabat} = \int pdV = -\Delta U$$

Furthermore, to compute the efficiency, we also need to compute the heat absorbed by the engine during the isochoric process $A \to B$.Since no work is done during the process, then the heat is also the change in the internal energy:

$$Q_{A\to B} = \Delta U$$

The internal energy of the system is related to the pressure via

$$\begin{aligned} dU &= TdS - pdV \\ \left(\frac{\partial U}{\partial V}\right)_T &= T\left(\frac{\partial S}{\partial V}\right)_T - p \\ &= T\left(\frac{\partial p}{\partial T}\right)_V - p \\ &= 0 \end{aligned}$$

where we used our Maxwell relations,

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

This suggests that the internal energy does not change with the volume of the gas. Hence,

$$\Delta U = C_V \Delta T$$

and total work from the Otto cycle is

$$\Delta W = W_{B\to C} + W_{D\to A} = C_V(T_B - T_C) + C_V(T_D - T_A)$$

Furthermore, the heat absorbed from the ischoric process $A \to B$ is

$$Q_{absorbed} = C_V(T_B - T_A)$$

and the efficiency of the system is

$$\eta = \frac{W}{Q_{absorbed}} = \frac{T_B - T_C + T_D - T_A}{T_B - T_A} = 1 - \frac{T_C - T_D}{T_B - T_A}$$

Now we must obtain relations between the 4 different temperatures of the Otto cycle. Consider the following thermodynamic relation:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{\left(\frac{\partial S}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_V} = -\frac{T}{C_V}\left(\frac{\partial S}{\partial V}\right)_T$$

Hence,

$$\left(\frac{\partial T}{\partial V}\right)_S = -\frac{T}{C_V}\left(\frac{\partial p}{\partial T}\right)_V = -\frac{nT}{C_V V}\left(R + \frac{n\alpha R}{V}\right)$$

Separating by variables and integrating, we get that

$$\ln T = -\frac{nR}{C_V}\ln V + \frac{n^2\alpha R}{C_V V} + g(S)$$

where $g(s)$ is some function of the entropy, S. From this relationship, we can easily show that

$$T = V^{-\frac{nR}{C_V}} e^{\frac{n^2\alpha R}{C_V V}} e^{g(S)} \qquad \Rightarrow \qquad TV^{\frac{nR}{C_V}} e^{-\frac{n^2\alpha R}{C_V V}} = \text{const.}$$

for an adiabatic process.

So, during the process $B \to C$ and $D \to A$,

$$T_A = T_D\left(\frac{V_2}{V_1}\right)^{\frac{nR}{C_V}} e^{\frac{n^2\alpha R}{C_V}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)} \qquad \text{and} \qquad T_B = T_C\left(\frac{V_2}{V_1}\right)^{\frac{nR}{C_V}} e^{\frac{n^2\alpha R}{C_V}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)}$$

and subtracting the two equations results in

$$T_B - T_A = (T_C - T_D)\left(\frac{V_2}{V_1}\right)^{\frac{nR}{C_V}} e^{\frac{n^2\alpha R}{C_V}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)}$$

Hence, the efficiency of the Otto engine is

$$\boxed{\eta = 1 - \left(\frac{V_2}{V_1}\right)^{-\frac{nR}{C_V}} e^{-\frac{n^2\alpha R}{C_V}\left(\frac{1}{V_1} - \frac{1}{V_2}\right)}}$$

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