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1) a. Let $\alpha = \frac{4e^2}{m^2c^2L^3}$

$$(\vec{S}_1+\vec{S}_2)^2 = S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2 \qquad \vec{S}_1+\vec{S}_2 = \vec{J}$$

$$(S_{1z}+S_{2z})^2 = S_{1z}^2 + S_{2z}^2 + 2S_{1z}S_{2z} \qquad S_{1z}+S_{2z} = J_z$$

$$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}(J^2 - S_1^2 - S_2^2)$$

$$3S_{1z}S_{2z} = \frac{3}{2}(J_z^2 - S_{1z}^2 - S_{2z}^2)$$

$$H = \frac{\alpha}{2}(J^2 - 3J_z^2) - \frac{\alpha}{2}(S_1^2 + S_2^2 - 3S_{1z}^2 - 3S_{2z}^2)$$

use basis $|j,m_j\rangle$ $\quad S_1^2 = S_2^2 = \hbar^2\cdot\frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4}\hbar^2$

$$S_{1z}^2 = S_{2z}^2 = \left(\pm\frac{\hbar}{2}\right)^2 = \frac{\hbar^2}{4}$$

$$\therefore H = \frac{\alpha}{2}(J^2 - 3J_z^2)$$

At $t=0$ $|j,m_j\rangle = |1,1\rangle$ $\quad (S_{1z} = S_{2z} = \frac{\hbar}{2})$

Since this is an eigenstate of the hamiltonian, time evolution simply introduces a phase factor that does not affect the probability. Thus at a later time, $S_{1z}+S_{2z} = \hbar$ with probability 1.

b. $|S_1\rangle = |\uparrow_{x1}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_1\rangle + |\downarrow_1\rangle)$

$|S_2\rangle = |\uparrow_{x2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_2\rangle + |\downarrow_2\rangle)$

$$|S_1\rangle\otimes|S_2\rangle = \frac{1}{2}(|\uparrow_1\uparrow_2\rangle + |\uparrow_1\downarrow_2\rangle + |\downarrow_1\uparrow_2\rangle + |\downarrow_1\downarrow_2\rangle)$$

$$|j,m_j\rangle = \frac{1}{2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle$$

$$|j,m_j\rangle_t = \frac{1}{2}e^{-iHt/\hbar}|1,1\rangle + \frac{1}{\sqrt{2}}e^{-iHt/\hbar}|1,0\rangle + \frac{1}{2}e^{-iHt/\hbar}|1,-1\rangle$$

$$H|1,1\rangle = \frac{\alpha}{2}(2\hbar^2 - 3\hbar^2) = -\frac{\alpha}{2}\hbar^2$$

$$H|1,0\rangle = \frac{\alpha}{2}(2\hbar^2 - 0) = \alpha\hbar^2$$

$$H|1,-1\rangle = \frac{\alpha}{2}(2\hbar^2 - 3(-\hbar)^2) = -\frac{\alpha}{2}\hbar^2$$

$$|j,m_j\rangle_t = \frac{1}{2}e^{i\alpha\hbar t/2}|1,1\rangle + \frac{1}{\sqrt{2}}e^{-i\alpha\hbar t}|1,0\rangle + \frac{1}{2}e^{i\alpha\hbar t/2}|1,-1\rangle$$

$$|1,1\rangle = |\uparrow_1\uparrow_2\rangle = \frac{1}{2}(|\uparrow_{x1}\rangle + |\downarrow_{x1}\rangle)(|\uparrow_{x2}\rangle + |\downarrow_{x2}\rangle)$$

$$= \frac{1}{2}(|\uparrow_{x1}\uparrow_{x2}\rangle + |\uparrow_{x1}\downarrow_{x2}\rangle + |\downarrow_{x1}\uparrow_{x2}\rangle + |\downarrow_{x1}\downarrow_{x2}\rangle)$$

$$|1,1\rangle = \frac{1}{2}|1,1\rangle_x + \frac{1}{\sqrt{2}}|1,0\rangle_x + \frac{1}{2}|1,-1\rangle_x$$

$$|1,-1\rangle = |\downarrow_1\downarrow_2\rangle = \frac{1}{2}(|\uparrow_{x1}\rangle - |\downarrow_{x1}\rangle)(|\uparrow_{x2}\rangle - |\downarrow_{x2}\rangle)$$

$$= \frac{1}{2}(|\uparrow_{x1}\uparrow_{x2}\rangle - |\uparrow_{x1}\downarrow_{x2}\rangle - |\downarrow_{x1}\uparrow_{x2}\rangle + |\downarrow_{x1}\downarrow_{x2}\rangle)$$

$$|1,-1\rangle = \frac{1}{2}|1,1\rangle_x - \frac{1}{\sqrt{2}}|1,0\rangle_x + \frac{1}{2}|1,-1\rangle_x$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_1 \downarrow_2\rangle + |\downarrow_1 \uparrow_2\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\Big[\left(|\uparrow_{x1}\rangle + |\downarrow_{x1}\rangle\right)\left(|\uparrow_{x2}\rangle - |\downarrow_{x2}\rangle\right) + \left(|\uparrow_{x1}\rangle - |\downarrow_{x1}\rangle\right)\left(|\uparrow_{x2}\rangle + |\downarrow_{x2}\rangle\right)\Big]$$

$$= \frac{1}{2\sqrt{2}}\left[2|\uparrow_{x1}\uparrow_{x2}\rangle - 2|\downarrow_{x1}\downarrow_{x2}\rangle\right)$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle_x + |1,-1\rangle_x\right)$$

$$\therefore |j,m_j\rangle_t = \frac{1}{2}e^{i\alpha\hbar t/2}\left[\frac{1}{2}|1,1\rangle_x + \cancel{\frac{1}{\sqrt{2}}|1,0\rangle_x} + \frac{1}{2}|1,-1\rangle_x\right]$$
$$+ \frac{1}{2}e^{i\alpha\hbar t/2}\left[\frac{1}{2}|1,1\rangle_x - \cancel{\frac{1}{\sqrt{2}}|1,0\rangle_x} + \frac{1}{2}|1,-1\rangle_x\right]$$
$$+ \frac{1}{\sqrt{2}}e^{-i\alpha\hbar t}\left[\frac{1}{\sqrt{2}}|1,1\rangle_x - \frac{1}{\sqrt{2}}|1,-1\rangle_x\right]$$
$$= \frac{1}{2}e^{i\alpha\hbar t/2}\left(|1,1\rangle_x + |1,-1\rangle_x\right)$$
$$+ \frac{1}{2}e^{-i\alpha\hbar t}\left(|1,1\rangle_x - |1,-1\rangle_x\right)$$
$$= e^{-i\alpha\hbar t/4}\Big[\frac{1}{2}\left(e^{3i\alpha\hbar t/4} + e^{-3i\alpha\hbar t/4}\right)|1,1\rangle_x$$
$$+ i\frac{1}{2i}\left(e^{3i\alpha\hbar t/4} - e^{-3i\alpha\hbar t/4}\right)|1,-1\rangle_x\Big]$$

$$|j,m_j\rangle_t = e^{-i\alpha\hbar t/4}\left[\cos\left(\tfrac{3}{4}\alpha\hbar t\right)|1,1\rangle_x + i\sin\left(\tfrac{3}{4}\alpha\hbar t\right)|1,-1\rangle_x\right]$$

$$\frac{3}{4}\alpha = \frac{3e^2}{m^2c^2L^3}$$

$\therefore$

$S_{1x}+S_{2x}$ value	Probability
$\hbar$	$\cos^2\left(\frac{3\hbar e^2}{m^2c^2L^3}t\right)$
$-\hbar$	$\sin^2\left(\frac{3\hbar e^2}{m^2c^2L^3}t\right)$
0	0

c. Classical dipoles will precess about the z axis with continuous spin values.

a) spins aligned with z-axis $S_{1z}+S_{2z}=\hbar$ with probability 1 as in the quantum case

b) $S_{1x}+S_{2x}=\hbar\cos\left(\frac{3\hbar e^2}{m^2c^2L^3}t\right)$. Unlike the quantum case the spins can take on continuous values as they precess.

b) In part (b) the spins align at $+\hat{z}$ or $-\hat{z}$, meaning that $S_{1x}+S_{2x}$ is measured to be 0 with probability 1.

At finite ~~but low~~ temperature the actual measurement of the classical spins should be expected to fluctuate