Problem M99Q.3

(a) Write

$$|\Psi\rangle = a\,|+\rangle + b\,|-\rangle\,,$$

so the total state of the three photons is

$$\begin{aligned}|\Psi_{tot}\rangle &= |\Psi\rangle \otimes \left|\Phi^{(a)}\right\rangle_{2,3} \\ &= \frac{1}{\sqrt{2}}\left[a\,|+,+,-\rangle_{1,2,3} - a\,|+,-,+\rangle_{1,2,3} + b\,|-,+,-\rangle_{1,2,3} - b\,|-,-,+\rangle_{1,2,3}\right].\end{aligned}$$

If Alice measures the state $\left|\Phi^{(a)}\right\rangle_{1,2}$, the state of the third photon is given by the projection

$$\left\langle\Phi^{(a)}\right|_{1,2}|\Psi_{tot}\rangle = \frac{1}{2}\left[-a\,|+\rangle - b\,|-\rangle\right] \sim a\,|+\rangle + b\,|-\rangle\,,$$

which is the original state $|\Psi\rangle$ (up to normalization and a global phase).

Similarly, if Alice measures the state $\left|\Phi^{(b)}\right\rangle_{1,2}$, then the third photon is in state

$$\left\langle\Phi^{(b)}\right|_{1,2}|\Psi_{tot}\rangle = \frac{1}{2}\left[-a\,|+\rangle + b\,|-\rangle\right] \sim a\,|+\rangle - b\,|-\rangle\,.$$

This can be transformed into $|\Psi\rangle$ via the unitary transformation σ_z.

(b) We have already shown this for outcomes a and b. For outcome c, Bob's photon is in state

$$a\,|-\rangle + b\,|+\rangle\,,$$

which can be transformed into the state $|\Psi\rangle$ by applying the unitary transformation σ_x.

For outcome d, Bob's photon is in state

$$a\,|-\rangle - b\,|+\rangle\,,$$

which can be transformed into $|\Psi\rangle$ by applying σ_z followed by σ_x.