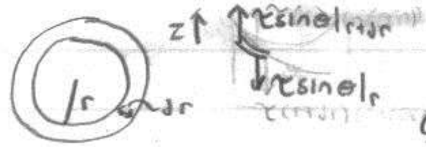
Seth Dorfman. Prelims December 27, 2005

Jan May 1999 CM

3) a. Consider a circular anulus:



$$0 = \rho(2\pi r\,dr)g + \tau(2\pi r)\sin(\theta(r)) - \tau(2\pi(r+dr))\sin(\theta(r+dr))$$

$$\frac{(r+dr)\sin(\theta(r+dr)) - r\sin(\theta(r))}{dr} = \frac{\rho g}{\tau} r$$

$$\frac{d}{dr}(r\sin(\theta(r))) = \frac{\rho g}{\tau} r$$

Assume τ large so θ small $\Rightarrow \sin\theta \approx \tan\theta = \frac{dz}{dr}$

$$\frac{d}{dr}\left(r\frac{dz}{dr}\right) = \frac{\rho g}{\tau} r$$

$r\frac{dz}{dr} = 0$ at $r=0 \Rightarrow c_1 = 0$

$$r\frac{dz}{dr} = \frac{\rho g}{2\tau} r^2 + c_1$$

$$\frac{dz}{dr} = \frac{\rho g}{2\tau} r$$

$z = 0$ at $r = 0 \Rightarrow c_2 = 0$

$$z(r) = \frac{\rho g}{4\tau} r^2 + c_2$$

$h = z(R) = \frac{\rho g}{4\tau} R^2$

Thus the center sags a height of $\frac{\rho g}{4\tau} R^2$ below the rim

b. $\rho(2\pi r\,dr)\ddot{z} = \tau(2\pi(r+dr))\sin(\theta(r+dr)) - \tau(2\pi r)\sin(\theta(r))$

$$\frac{d}{dr}(r\sin\theta(r)) = \frac{\rho r}{\tau}\ddot{z} \qquad \sin\theta \approx \frac{dz}{dr}$$

$$\frac{d}{dr}\left(r\frac{dz}{dr}\right) = \frac{\rho r}{\tau}\frac{\partial^2 z}{\partial t^2} \qquad z(r;t) = Z(r)e^{-i\omega t}$$

$$\frac{dZ}{dr} + r\frac{d^2 Z}{dr^2} = -\omega^2\frac{\rho r}{\tau}Z \qquad \text{Let } \alpha^2 = \omega^2\frac{\rho}{\tau}$$

$$\frac{d^2 Z}{dr^2} + \frac{1}{r}\frac{dZ}{dr} + \alpha^2 Z = 0 \qquad \text{Let } f(x) = \frac{Z(r)}{R}$$

$$\alpha^2\frac{\partial^2}{\partial x^2}(fR) + \frac{\alpha}{x}\alpha\frac{\partial}{\partial x}(fR) + \alpha^2(fR) = 0 \qquad x = \alpha r \qquad \frac{\partial}{\partial r} = \alpha\frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial x^2}f + \frac{1}{x}\frac{\partial f}{\partial x} + f = 0$$

The solutions to this equation are Bessel functions.

For the lowest order frequency, $f(x) = J_0(x)$

$Z(r) = R\,J_0(\alpha r)$

B.C. $Z(r) = 0$ at $r = R$

Since we are given $J_0(0.766\pi) = 0$ as the first zero, the lowest order mode has $\alpha R = 0.766\pi$

$$\omega\sqrt{\frac{\rho}{\tau}} = \frac{0.766\pi}{R}$$

$$2\pi f = \omega = 0.766\pi\frac{\sqrt{\rho}}{R\sqrt{\tau}}$$

$$f = 0.383\frac{1}{R}\sqrt{\frac{\rho}{\tau}}$$