M99M.3

a) Consider a ring of width dr. $F = ma$ gives:

$$0 = -2\pi dr \rho g + \tau 2\pi (r + dr) \sin(\theta(r + dr)) - \tau 2\pi r \sin\theta \tag{1}$$

Where $\theta(r + dr)$ is evaluated at $(r + dr)$. Rearranging this gives

$$\frac{\rho g r}{\tau} = \frac{r \sin(\theta(r + dr)) - r \sin\theta}{dr} + \sin\theta(r + dr). \tag{2}$$

Assuming θ is small, we can ignore the second term on the right hand side. Next, in the limit $dr \to 0$, the remaining right hand side term is equal to $\frac{d(r \sin\theta)}{dr}$ (by definition of a derivative).

We can then approximate $\sin\theta$ as $\tan\theta = \frac{dz}{dr}$ as θ is small, giving

$$\frac{d}{dr}\frac{rdz}{dr} = \rho g r \tau. \tag{3}$$

Integrating twice gives z as a function of r:

$$z = \frac{\rho g r^2}{4\tau} + c_1 \log r + c_2, \tag{4}$$

and $c_1 = 0$ as $z(0)$ must be finite (at the center of the membrane), and $z(R) = 0$ (at the edges of the drum) gives $c_2 = -\frac{\rho g R^2}{4\tau}$. The membrane thus sags a vertical distance $z(0)$ or $-\frac{\rho g R^2}{4\tau}$ which is confirmed by the Russian solutions.

b) Ignoring gravity, $F = ma$ gives

$$2\pi r\partial r\rho = 2\pi(r+\partial r)\tau\sin\theta(r+\partial r) - 2\pi r\tau\sin\theta. \tag{5}$$

Proceeding as before,

$$\frac{\rho r}{\tau}\partial_t^2 z = \partial_r(r\sin\theta) = \partial_r(r\partial_r z) = \partial_r z + r\partial_r^2 z \tag{6}$$

Approximating the membrane as a simple harmonic oscillator, let $\partial_t^2 z = -\omega^2 z$. Rearranging, this gives

$$\partial_r^2 z + \frac{1}{r}\partial_r z + \frac{\rho\omega^2}{\tau} z. \tag{7}$$

To make this resemble the Bessel function in the hint, define $\lambda = \frac{\rho\omega^2}{\tau}$ and let $x = \sqrt{\lambda}r$ (thus $\partial_r = \sqrt{\lambda}\partial_x$). The boundary condition $z(R) = 0$ is equivalent to requiring that $z(\lambda R) = 0$. From the hint, $z(\sqrt{\lambda}R) = 0$ when $\sqrt{\lambda}R = 0.766\pi$. Plugging in the definition of λ and solving gives $\omega = 0.766\pi\sqrt{\frac{\tau}{\rho}}$ as the lowest vibrational frequency.

One thought on “M99M.3”

December 2, 2013 at 10:41 pm

Correct. You've lost R in your final answer for ω though.
