M99M.3

a) Consider a ring of width $dr. \quad F = ma$ gives:

$$0 = -2\pi dr \rho g + \tau 2\pi (r + dr) \sin(\theta(r + dr)) - \tau 2\pi r \sin \theta$$  \quad (1)

Where $\theta(r + dr)$ is evaluated at $(r + dr)$. Rearranging this gives

$$\frac{\rho gr}{\tau} = \frac{r \sin(\theta(r + dr)) - r \sin \theta}{dr} + \sin(\theta(r + dr)).$$  \quad (2)

Assuming $\theta$ is small, we can ignore the second term on the right hand side. Next, in the limit $dr \to 0$, the remaining right hand side term is equal to $\frac{d(r \sin \theta)}{dr}$ (by definition of a derivative).

We can then approximate $\sin \theta$ as $\tan \theta = \frac{dz}{dr}$ as $\theta$ is small, giving

$$\frac{d}{dr} \frac{rdz}{dr} = \rho gr \tau.$$  \quad (3)

Integrating twice gives $z$ as a function of $r$:

$$z = \frac{\rho gr^2}{4\tau} + c_1 \log r + c_2,$$  \quad (4)

and $c_1 = 0$ as $z(0)$ must be finite (at the center of the membrane), and $z(R) = 0$ (at the edges of the drum) gives $c_2 = -\frac{\rho g R^2}{4\tau}$. The membrane thus sags a vertical distance $z(0)$ or $-\frac{\rho g R^2}{4\tau}$ which is confirmed by the Russian solutions.

b) Ignoring gravity, $F = ma$ gives
Proceeding as before,

\[
\frac{\rho r}{\tau} \frac{\partial^2}{\partial_t^2} z = \partial_r (r \sin \theta) = \partial_r (r \partial_r z) = \partial_r z + r \partial_r^2 z
\]  

(6)

Approximating the membrane as a simple harmonic oscillator, let \( \partial^2_t z = -\omega^2 z \). Rearranging, this gives

\[
\partial_r^2 z + \frac{1}{t} \partial_r z + \frac{\rho \omega^2}{\tau} z.
\]  

(7)

To make this resemble the Bessel function in the hint, define \( \lambda = \frac{\omega^2}{\tau} \) and let \( x = \sqrt{\lambda r} \) (thus \( \partial_r = \sqrt{\lambda} \partial_x \)). The boundary condition \( z(R) = 0 \) is equivalent to requiring that \( z(\lambda R) = 0 \). From the hint, \( z(\sqrt{\lambda} R) = 0 \) when \( \sqrt{\lambda} R = 0.766\pi \). Plugging in the definition of \( \lambda \) and solving gives \( \omega = 0.766\pi \sqrt{\frac{r}{\rho}} \) as the lowest vibrational frequency.

One thought on “M99M.3”

Correct. You've lost \( R \) in your final answer for \( \omega \) though.