M99M.3

We use $u(r,\phi)$ to denote the deformation of the drum (the membrane). The force F on a small fraction of the drum at (r,ϕ) is

$$\delta\vec{F} = \delta m\vec{g} + \vec{P} \tag{1}$$

where $\vec{P}$ is the force from the tension of the drum.

$$\delta F_{\perp} = \delta mg + P_{\perp} \tag{2}$$
$$\delta F_{\|} = P_{\|} \tag{3}$$

where $\perp$ denotes the vertical direction, and $\|$ denotes the horizontal direction.\\
Consider the tension on a small fraction of the drum with vertice at $(x,y),(x+\delta x,y),(x+\delta x,y+\delta y),(x,y+\delta y)$, where $x=r\cos\phi, y=r\sin\phi$. Assume the displacement of this piece is $u(x,y)$. The angle between $u(x,y)$ and (x,y) plane has a projection $\theta=\arctan(\frac{\partial u}{\partial x})\sim\frac{\partial u}{\partial x}$ on the (y,z) plane.\\
Therefore we should have the tension on the x direction and on the vertical direction as:
$P_{\perp}=(\tau\delta y\sin\theta)_{x+\delta x}-(\tau\delta y\sin\theta)_x\sim(\tau\delta y\theta)_{x+\delta x}-(\tau\delta y\theta)_x=(\tau\delta y\theta)_{x+\delta x}$
$-(\tau\delta y\theta)_x$
$P_{\|}=(\tau\delta y\cos\theta)_{x+\delta x}-(\tau\delta y\cos\theta)_x\sim(\tau\delta y(-\frac{1}{2})\theta^2)_{x+\delta x}-(\tau\delta y(-\frac{1}{2})\theta^2)_x$
consider $\theta=\frac{\partial u}{\partial x}$, and the same picture for the y direction (Assume the projection of the intersection angle on x,z plane is $\theta'=\frac{\partial u}{\partial y}$), we can get:

$$P_{\perp} = \tau\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_{x}\right] + \tau\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_{y}\right]$$
$$= \tau\delta x\delta y\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$
$$= \tau\delta A\nabla^2 u \quad (4)$$
$$P_{\parallel} = \tau\delta y\frac{1}{2}\left[-\left(\frac{\partial u}{\partial x}\right)^2_{x+\delta x} + \left(\frac{\partial u}{\partial x}\right)^2_{x}\right] + \tau\delta x\frac{1}{2}\left[-\left(\frac{\partial u}{\partial y}\right)^2_{y+\delta y} + \left(\frac{\partial u}{\partial y}\right)^2_{y}\right] \quad (5)$$

where $\delta A = \delta x\delta y$ is the area of the small fraction of the drum.
Since we consider small deformation, $\frac{\partial u}{\partial x_i}$ is condiered small, that is, $\frac{\partial u}{\partial x_i} \ll 1$, therefore, $P_{\parallel} \ll P_{\perp}$, so we consider only the vertical force on the drum, getting:

$$\delta F = \delta mg + \tau\delta A\nabla^2 u \quad (6)$$
$$\Rightarrow \delta m\frac{\partial^2 u}{\partial t^2} = \delta F = \delta mg + \tau\delta A\nabla^2 u \quad (7)$$
$$\Rightarrow \frac{\partial^2 u}{\partial t^2} - \frac{\tau}{\rho}\nabla^2 u = g \quad (8)$$

where $\rho = \frac{\delta m}{\delta A}$ is mass per unit area.

\\

(a) The center sag below the rim is displacement at the center $u|_{r=0}$.

Equilibrium state, so we have

$$\frac{\partial u}{\partial t} = 0. \quad (9)$$

From the azimuthal symmetry we have

$$\frac{\partial u}{\partial \phi} = 0. \quad (10)$$

We also have the edge of the drum is the level of the rim, and the center does not sag to infitely low level:

$$u|_{r=R} = 0 \quad (11)$$
$$u|_{r=0} = finite \quad (12)$$

from (8), (9), (10) and the form of u in cylindrical geometry we have:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\rho g}{\tau}=0 \tag{13}$$

$$\Rightarrow\ u=A_1r^2+B_1\ln r+C_1 \tag{14}$$

From (13) we have $A_1=-\rho g/4\tau$. From (12) we have $B_1=0$. From (11) we have $C_1=\frac{\rho g}{4\tau}R^2$.
Therefore we have

$$u=-\frac{\rho g}{4\tau}r^2+\frac{\rho g}{4\tau}R^2 \tag{15}$$

$$u\big|_{r=0}=\frac{\rho g}{4\tau}R^2 \tag{16}$$

\\
\\
(b) Assume $u=R(r)\Phi(\phi)T(t)$. From (8) and $g=0$, we have

$$\frac{T''}{T}-\frac{\tau}{\rho}\left[\frac{(rR')'}{rR}+\frac{\Phi''}{r^2\Phi}\right]=0 \tag{17}$$

$$R(r=0)=finite \tag{18}$$

$$R(r=R)=0 \tag{19}$$

$$\Phi(0)=\Phi(2\pi) \tag{20}$$

$$\Phi'(0)=\Phi'(2\pi) \tag{21}$$

where $X(x)'=\frac{dX}{dx}$ denotes first order derivative, and $X(x)''=\frac{d^2X}{dx^2}$ denotes second order derivative.
From (17) we have:

$$T''+\omega^2T=0\ \ \Rightarrow T=A_\omega e^{i\omega t}+B_\omega e^{-i\omega t} \tag{22}$$

$$\Phi''+m^2\Phi=0\ \ \Rightarrow \Phi_m=\cos(m\phi)+C_m\sin(m\phi) \tag{23}$$

$$(20)\&(21)\Rightarrow\ m\in Z$$

$$\frac{(rR_m')'}{(rR_m)}+\left(\frac{\omega_i^2\rho}{\tau}-\frac{m^2}{r^2}\right)=0 \tag{24}$$

$$\Rightarrow\ \ R_m=J_m(kr)+D_mN_m(kr) \tag{25}$$

where $A,B,C\&D$ are parmameters determined by the initial condition. $k^2=\omega^2\rho/\tau$.\\
From (18), we get $D_m=0$ for any m.\\
From (19), we get

$$J_m(kR) = 0 \quad \Rightarrow \tag{26}$$

$$k = k_{im} = \frac{\mu_{im}}{R} \quad \Rightarrow \omega_{im} = \frac{\mu_{im}}{R}\sqrt{\frac{\tau}{\rho}} \tag{27}$$

where μ_{im} is the i^{th} 0 point of the m^{th} order Bessel function $J_m(x)$, s.t. For $m = 0, 1, 2...$, $J_m(\mu_{im}) = 0, \ (i = 0, 1, 2...)$.
In the problem, the smallest zero point $\mu_{00} = 0.766\pi$ is given. From (27) we have

$$\omega_{00} = 0.766\pi\sqrt{\frac{\tau}{\rho R^2}} \tag{28}$$

Therefore the lowest vibrational frequency f would be

$$f = \omega_{00}/2\pi = 0.383\sqrt{\frac{\tau}{\rho R^2}} \tag{29}$$

One thought on “M99M.3”

October 17, 2013 at 5:18 pm

Very good. I have no major comments.
Just two small ones:
1) in the first line of your solution your write $u(r, z, \phi)$, why do you include z?
2) Equation (5) looks a bit suspicious. Even though the exact expression in (5) doesn't make any impact on your solution, it's better to reexamine it (and maybe provide some explanation?).
