M99M.3

Solution to M99M.3 — Drum Head

1. The force on a ring with width dr:

$$F = 0 = -2\pi r\rho g dr + \tau 2\pi(r + dr)sin(\theta(r + dr)) - \tau 2\pi r sin(\theta) \qquad (1)$$

   Manipulate a little.

$$\frac{d(rsin(\theta))}{dr} = \frac{\rho g r}{\tau} \qquad (2)$$

   Where $sin(\theta) = tan(\theta) = \frac{dz}{dr}$, because the tension is large and angles are small. Then I got

$$\frac{d}{dr}(r\frac{dz}{dr}) = \frac{\rho g r}{\tau} \qquad (3)$$

   After integrating once, I got

$$r\frac{dz}{dr} = \frac{\rho g r^2}{2\tau} + C_1 \qquad (4)$$

   Where C_1 is a constant. Integrate again, I got

$$z = \frac{\rho g r}{2\tau} + C_1 ln(r) + C_2 \qquad (5)$$

   Now, consider the boundary conditions we have. $z(0)$ must be finite, so $C_1 = 0$. $z(R) = 0$, so $C_2 = -\frac{\rho g R^2}{4\tau}$. Finally, I got $z(0) = -\frac{\rho g R^2}{4\tau}$.
2. Apply Newton's law to get the equation of motion.

$$2\pi r\rho\partial r\partial_t^2 z = \tau 2\pi(r + \partial r)sin(\theta(r + \partial r)) - \tau 2\pi r sin(\theta) \qquad (6)$$

   Consequently

$$\frac{r\rho}{\tau}\partial_t^2 z = \partial_r(r\partial_r z) = \partial_r z + r\partial_r^2 z \tag{7}$$

I expect z to be in form of $z(r,t) = z(r)\exp(-i\omega t)$ So $\partial_t^2 z = -\omega^2 z$. Define $\lambda = \frac{\omega^2\rho}{\tau}$.

$$\partial_r^2 z + \frac{1}{r}\partial_r z + \lambda z = 0 \tag{8}$$

To get standard partial form of Bessel function, I define $x = \sqrt{\lambda}r$. Now we have

$$\partial_x^2 z + \frac{1}{x}\partial_x z + z = 0 \tag{9}$$

The solution to the Bessel function is $z(x) = J_0(x)$. Write x in terms of r $z(\sqrt{\lambda}r) = J_0(\sqrt{\lambda}r)$. Consider the boundary condition, we have $J_0(\sqrt{\lambda}R) = 0$. $J_0(0.766\pi) = 0$ is given by the problem. So I got $\sqrt{\lambda}R = 0.766\pi$. Write λ in terms of ω, I finally got $\omega = \frac{0.766\pi}{R}\sqrt{\frac{\tau}{\rho}}$.

One thought on “M99M.3”

October 26, 2013 at 10:07 pm

Good, everything is correct.
I don't really like your notation ∂r for the infinitely small change in r, but otherwise it's OK.