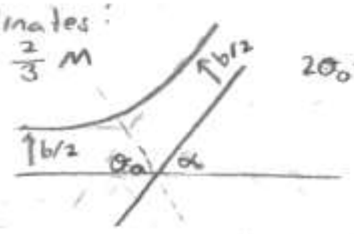
May 1999 CM #2

Using center of mass coordinates:

$\mu = \frac{m_1 m_2}{m_1+m_2} = \frac{M(2M)}{M+2M} = \frac{2}{3}M$



$2\theta_0 + \alpha = \pi$

$M_{tot} = 3M$

$U = \frac{Mv + 2M(0)}{3M} = \frac{1}{3}v$

$\mu v_\mu = 2Mv$

$\frac{2}{3}M v_\mu = \frac{2}{3}M v \Rightarrow v_\mu = v$

$E_{tot} = \frac{1}{2}M_{tot}U^2 + \frac{1}{2}\mu v^2 + \frac{L^2}{2\mu r^2} + \frac{a}{r^2} = \frac{1}{2}M_{tot}U^2 + E_{cm}$

$E_{cm} = \frac{1}{2}\mu v^2 + \frac{1}{r^2}\left(\frac{L^2}{2\mu} + a\right)$

$v^2 = \frac{2}{\mu}\left(E_{cm} - \frac{1}{r^2}\left(\frac{L^2}{2\mu}+a\right)\right)$ $\qquad L = \mu r^2\dot\theta$

$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left(E_{cm} - \frac{1}{r^2}\left(\frac{L^2}{2\mu}+a\right)\right)}$

$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$

$\frac{dr}{d\theta} = \sqrt{\frac{2}{\mu}\left(E_{cm} - \frac{1}{r^2}\left(\frac{L^2}{2\mu}+a\right)\right)}\,\frac{\mu r^2}{L}$ $\qquad u = \frac{1}{r} \quad du = \frac{-1}{r^2}dr$

$\frac{du}{d\theta} = -\frac{\mu}{L}\sqrt{\frac{2}{\mu}\left(E_{cm} - u^2\left(\frac{L^2}{2\mu}+a\right)\right)}$ $\qquad du = -u^2 dr$

$\frac{du}{d\theta} = -\frac{\sqrt{2\mu}}{L}\sqrt{E_{cm} - K^2u^2}$ $\qquad$ Let $K^2 = \frac{L^2}{2\mu} + a$

$\frac{d\phi}{d\theta}\cos\phi\frac{\sqrt{E_{cm}}}{K} = -\frac{\sqrt{2\mu}}{L}\sqrt{E_{cm}}\sqrt{1-\sin^2\phi}$ $\qquad$ Let $\frac{K}{\sqrt{E_{cm}}}u = \sin\phi$

$\frac{d\phi}{d\theta} = -\frac{K}{L}\sqrt{2\mu}$ $\qquad \frac{K}{\sqrt{E_{cm}}}du = \cos\phi\, d\phi$

$\phi = -\frac{K}{L}\sqrt{2\mu}\,\theta + C_1$

$\sin\phi = -\sin\left(\frac{K}{L}\sqrt{2\mu}\,\theta\right) + C_1$

$\frac{K}{\sqrt{E_{cm}}}\frac{1}{r} = -\sin\left(\frac{K}{L}\sqrt{2\mu}\,\theta + C_1\right)$ $\qquad \theta = 0, 2\theta_0 \Rightarrow r = \infty \quad C_1 = \pi$

$0 = +\sin\left(\frac{K}{L}\sqrt{2\mu}\,2\theta_0 + \pi\right)$

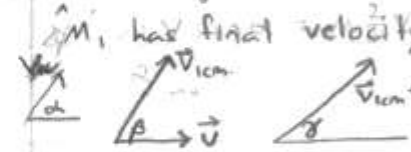
$\frac{K}{L}\sqrt{2\mu}\,(2\theta_0) = \pi$

$\pi - \alpha = \frac{L}{K}\frac{\pi}{\sqrt{2\mu}}$

$\alpha = \pi\left(1 - \frac{L}{K}\frac{1}{\sqrt{2\mu}}\right)$

Transforming the final momentum back to the lab frame:

$M_1$ has final velocity $\frac{2}{3}v$ in CM frame



$v_{1cm}\sin\beta = |\vec v_{1cm} + \vec U|\sin\gamma$

$\frac{2}{3}v\sin\beta = \left[\left(\frac{2}{3}v\right)^2 + 2\cdot\frac{2}{3}v\cdot\frac{1}{3}v\cos\beta + \left(\frac{1}{3}v\right)^2\right]^{1/2}\sin\gamma$

$\frac{2}{3}\sin\beta = \left[\frac{5}{9} + \frac{4}{9}\cos\beta\right]^{1/2}\sin\gamma$

$\frac{2}{3}M$

$\mu v\sin\alpha = \mu\left(\frac{2}{3}v\right)\sin\beta$

$\Rightarrow \sin\alpha = \sin\beta$

$\sin\gamma = \frac{2\sin\alpha}{\sqrt{5+4\cos\alpha}} = \frac{2\sin\left(\pi\frac{L}{K}\frac{1}{\sqrt{2\mu}}\right)}{\sqrt{5+4\cos\left(\pi\frac{L}{K}\frac{1}{\sqrt{2\mu}}\right)}}$

∴ the deflection angle is given by $\sin\gamma = \frac{2\sin\left(\pi\left(1+\frac{4Ma}{3L^2}\right)^{-1/2}\right)}{\sqrt{5+4\cos\left(\pi\left(1+\frac{4Ma}{3L^2}\right)^{-1/2}\right)}}$ $\qquad L = \frac{1}{3}mvb$