Electromagnetism

M99-3 Dielectric Cylinder

long so no z dependence

no free charge, so

$$V(r,\theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} [A_n r^n + B_n r^{-n}][C_n \cos(n\theta) + D_n \sin(n\theta)]$$

there's also an external potential $V_e(r,\theta) = -E_0 x = -E_0 r\cos(\theta)$
only in the outer region

arb set $A_0 = 0$

now V must be finite at $r=0$

so $V_{in}(r,\theta) = \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$
diff consts from general

far from the cylinder $\vec{E} \to E_0\hat{x}$, so the contribution to V due to the cylinder must go to zero $(A_n, A_0, B_0) \to 0$

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \sum_{n=1}^{\infty} r^{-n}[C_n\cos(n\theta) + D_n \sin(n\theta)]$$

must have $V_{in}(R,\theta) = V_{out}(R,\theta)$

$$\underbrace{\varepsilon_{out}}_{1} \frac{\partial V_{out}}{\partial r}(R,\theta) - \underbrace{\varepsilon_{in}}_{\varepsilon} \frac{\partial V_{in}}{\partial r}(R,\theta) = 4\pi \cancel{\sigma_f}^{\,0}$$

$$\frac{\partial V_{out}}{\partial r} = -E_0\cos\theta - \sum_{n=1}^{\infty} n\, r^{-(n+1)}[C_n\cos(n\theta) + D_n\sin(n\theta)]$$
$$\frac{\partial V_{in}}{\partial r} = \sum_{n=1}^{\infty} n\, r^{n-1}[A_n\cos(n\theta) + B_n \sin(n\theta)]$$

now $\cos(n\theta)$ & $\sin(n\theta)$ linearly ind. of $\cos(m\theta)$ & $\sin(m\theta)$ $m \neq n$

let's look at $n=1$.

$$R[A_1\cos(\theta) + B_1\sin\theta] = -E_0 R\cos\theta + \frac{1}{R}[C_1\cos\theta + D_1\sin\theta]$$
$$\varepsilon(A_1\cos(\theta) + B_1\sin\theta) = -E_0\cos\theta - \frac{1}{R^2}[C_1\cos\theta + D_1 \sin\theta$$

so $R(A_1 \cos\theta + B_1 \sin\theta) = -E_0 R\cos\theta + \frac{1}{R}(C_1\cos\theta + D_1 \sin\theta)$

$\varepsilon R(A_1\cos\theta + B_1\sin\theta) = -E_0 R\cos\theta - \frac{1}{R}(C_1\cos\theta + D_1\sin\theta)$

sum them

$(1+\varepsilon)R(A_1\cos\theta + B_1\sin\theta) = -2E_0 R\cos\theta$

$B_1 = 0$

$\boxed{A_1 = -\frac{2E_0}{1+\varepsilon}}$

since $B_1 = 0$, $D_1 = 0$ no $\sin\theta$ dependence

so $-\frac{2E_0}{1+\varepsilon} R \cancel{\cos\theta} = -E_0 R \cancel{\cos\theta} + \frac{1}{R} C_1 \cancel{\cos\theta}$

so $\boxed{C_1 = \left(E_0 - \frac{2E_0}{1+\varepsilon}\right) R^2}$

now look at $n=2$ (no more E_0 term since that was in $n=1$)

$R^2(A_2\cos(2\theta) + B_2\sin(2\theta)) = \frac{1}{R^2}(C_2\cos(2\theta) + D_2\sin(2\theta))$

$2\varepsilon R^2(A_2\cos(2\theta) + B_2\sin(2\theta)) = -\frac{2}{R^2}(C_2\cos(2\theta) + D_2\sin(2\theta))$

so coeffs must match $A_2 R^2 = \frac{C_2}{R^2}$

but $A_2(2\varepsilon R^2) = -\frac{2C_2}{R^2}$ $A_2 = C_2 = 0$

same goes for $B_2 = D_2 = 0$

and likewise for all other n

thus we only have $n = 1$

$$V(r,\theta) = \begin{cases} \left(-\frac{2E_0}{1+\varepsilon}\right) r\cos\theta & r \leq R \\ -E_0 r\cos\theta + \left(E_0 - \frac{2E_0}{1+\varepsilon}\right)\frac{R^2}{r}\cos\theta & r > R \end{cases}$$

$E = -\nabla V$

$E_{in} = \frac{2E_0}{1+\varepsilon}\cos\theta\,\hat{r} - \frac{2E_0}{1+\varepsilon}\sin\theta\,\hat{\theta}$ $r \leq R$

$= \frac{2E_0}{1+\varepsilon}\hat{x}$

$E_{out} = \left[E_0\cos\theta + \left(E_0 - \frac{2E_0}{1+\varepsilon}\right)\frac{R^2}{r^2}\cos\theta\right]\hat{r} + \left[-E_0\sin\theta + \left(E_0 - \frac{2E_0}{1+\varepsilon}\frac{R^2}{r^2}\sin\theta\right)\right]\hat{\theta}$