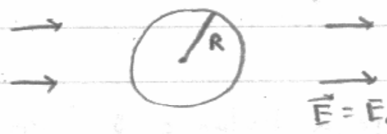
May 1999 EM #3



$\vec{E} = E_0 \hat{x}$

$\phi|_{r=\infty} = -E_0 x = -E_0 r\cos\theta$

At $r = R$ $\quad E^{\parallel}$ continuous

$\quad\quad\quad\quad D^{\perp}$ continuous

$$\varepsilon E^{\perp}|_{R^-} = E^{\perp}|_{R^+} \Rightarrow \varepsilon \frac{\partial\phi}{\partial r}\Big|_{R^-} = \frac{\partial\phi}{\partial r}\Big|_{R^+}$$

$$E^{\parallel}|_{R^-} = E^{\parallel}|_{R^+} \Rightarrow \frac{\partial\phi}{\partial\theta}\Big|_{R^-} = \frac{\partial\phi}{\partial\theta}\Big|_{R^+}$$

The solution to Poisson's equation in the two regions are given by:

$$\phi|_{r<R} = \sum_n (A_n\cos(k_n\theta) + B_n\sin(k_n\theta))\, r^{k_n}$$

$$\phi|_{r>R} = \sum_n (C_n\cos(k_n\theta) + D_n\sin(k_n\theta))\, \frac{1}{r^{k_n}} - E_0 r\cos\theta$$

$$\frac{\partial\phi}{\partial r}\Big|_{R^-} = \sum_n (A_n\cos(k_n\theta) + B_n\sin(k_n\theta))\, k_n r^{k_n-1}$$

$$\frac{\partial\phi}{\partial r}\Big|_{R^+} = \sum_n (C_n\cos(k_n\theta) + D_n\sin(k_n\theta))(-k_n)\, \frac{1}{r^{k_n+1}} - E_0\cos\theta$$

Matching terms:

$k_n = 1 \quad (A_1\cos(\theta) + B_1\sin(\theta))\,\varepsilon = -E_0\cos\theta$

$$\quad\quad - (C_1\cos\theta + D_1\sin\theta)\frac{1}{R^2}$$

$$\Rightarrow \varepsilon A_1 = -E_0 - \frac{C_1}{R^2} \qquad \varepsilon B_1 = \frac{-D_1}{R^2}$$

$k_n \neq 1 \quad \varepsilon(A_n\cos(k_n\theta) + B_n\sin(k_n\theta))\, k_n R^{k_n-1}$

$$= ((C_n\cos(k_n\theta) + D_n\sin(k_n\theta))\cdot(-\cancel{k_n})\,\frac{1}{R^{k_n+1}}$$

$$\Rightarrow \varepsilon A_n R^{k_n-1} = -C_n \frac{1}{R^{k_n+1}} \qquad \varepsilon B_n R^{k_n-1} = -D_n \frac{1}{R^{k_n+1}}$$

$$\varepsilon A_n R^{2k_n} = -C_n \qquad \varepsilon B_n R^{2k_n} = -D_n$$

$$\frac{\partial\phi}{\partial\theta}\Big|_{R^-} = \sum_n (-A_n k_n\sin(k_n\theta) + B_n k_n\cos(k_n\theta))\, r^{k_n}$$

$$\frac{\partial\phi}{\partial\theta}\Big|_{R^+} = \sum_n (-C_n k_n\sin(k_n\theta) + D_n k_n\cos(k_n\theta))\, \frac{1}{r^{k_n}} + E_0 r\sin\theta$$

Matching terms:

$k_n = 1 \quad (-A_1\sin\theta + B_1\cos\theta)R = (-C_1\sin\theta + D_1\cos\theta)\frac{1}{R} + E_0 R\sin\theta$

$$\Rightarrow -A_1 R = \frac{-C_1}{R} + E_0 R \qquad -B_1 R = \frac{-D_1}{R}$$

$k_n \neq 1 \quad (-A_n\cancel{k_n}\sin(k_n\theta) + B_n\cancel{k_n}\cos(k_n\theta))\, R^{k_n}$

$$= (-C_n\cancel{k_n}\sin(k_n\theta) + D_n\cancel{k_n}\cos(k_n\theta))\,\frac{1}{R^{k_n}}$$

$$\Rightarrow -A_n R^{k_n} = -C_n\frac{1}{R^{k_n}} \qquad B_n R^{k_n} = D_n \frac{1}{R^{k_n}}$$

$$A_n R^{2k_n} = C_n \qquad B_n R^{2k_n} = D_n$$

if $k_n \neq 1$, $\varepsilon \neq -1$ $\quad A_n = B_n = C_n = D_n = 0$ (from above)

For $k_n = 1$, $\varepsilon \neq -1$ $\quad B_1 = D_1 = 0$

$$A_1 = \frac{1}{\varepsilon}\left(-E_0 - \frac{C_1}{R^2}\right) = \frac{C_1}{R^2} - E_0$$

$$E_0\left(-\frac{1}{\varepsilon} + 1\right) = \frac{C_1}{R^2}\left(1 + \frac{1}{\varepsilon}\right)$$

(see next page)

Seth Dorfman Prelims December 28, 2005

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#3 (continued)

$E_0\left(\frac{\varepsilon-1}{\varepsilon}\right) = \frac{C_1}{R^2}\left(\frac{\varepsilon+1}{\varepsilon}\right)$ $\qquad A_1 = \frac{C_1}{R^2} - E_0$

$C_1 = E_0 R^2\left(\frac{\varepsilon-1}{\varepsilon+1}\right)$ $\qquad A_1 = E_0\left(\frac{\varepsilon-1}{\varepsilon+1}\right) - E_0\left(\frac{\varepsilon+1}{\varepsilon+1}\right)$

$A_1 = -E_0\left(\frac{2}{\varepsilon+1}\right)$

$$\phi|_{r<R} = -E_0\left(\frac{2}{\varepsilon+1}\right)\cos(\theta)\, r$$

$$\phi|_{r>R} = E_0 R^2\left(\frac{\varepsilon-1}{\varepsilon+1}\right)\cos\theta\,\frac{1}{r} - E_0\, r\cos\theta$$

$$\vec{E} = -\vec{\nabla}\phi = -\frac{\partial\phi}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta}$$

$$\vec{E}|_{r<R} = E_0\left(\frac{2}{\varepsilon+1}\right)\cos\theta\,\hat{r} - E_0\left(\frac{2}{\varepsilon+1}\right)\sin\theta\,\hat{\theta}$$

$$\boxed{\vec{E}|_{r<R} = E_0\left(\frac{2}{\varepsilon+1}\right)\hat{x}}$$

$$\vec{E}|_{r>R} = E_0 R^2\left(\frac{\varepsilon-1}{\varepsilon+1}\right)\cos\theta\,\frac{1}{r^2}\hat{r} + E_0 R^2\left(\frac{\varepsilon-1}{\varepsilon+1}\right)\sin\theta\,\frac{1}{r^2}\hat{\theta}$$
$$+ E_0\cos\theta\,\hat{r} - E_0\sin\theta\,\hat{\theta}$$

$$\boxed{\vec{E}|_{r>R} = E_0\left(\frac{R}{r}\right)^2\left(\frac{\varepsilon-1}{\varepsilon+1}\right)(\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) + E_0\hat{x}}$$