(a) Since $f < 0$, the minimum-energy spin configuration will be the one with the greatest value of $\hat{S}_3^{(a)} - \hat{S}_3^{(b)}$. This occurs when we have eigenstates of $\hat{S}_3$, specifically with $m_a = s_a, m_b = -s_b$. We may compute the expectation value as follows:

$$\hat{S}^2 = (\hat{\mathbf{S}}^{(a)} + \hat{\mathbf{S}}^{(b)})^2 \tag{1}$$
$$\hat{S}^2 = \hat{S}^{(a)2} + \hat{S}^{(b)2} + 2\hat{\mathbf{S}}^{(a)} \cdot \hat{\mathbf{S}}^{(b)} \tag{2}$$
$$\hat{S}^2 = \hat{S}^{(a)2} + \hat{S}^{(b)2} + 2(\hat{S}_1^{(a)}\hat{S}_1^{(b)} + \hat{S}_2^{(a)}\hat{S}_2^{(b)}) + 2\hat{S}_3^{(a)}\hat{S}_3^{(b)} \tag{3}$$
$$\langle\hat{S}^2\rangle = \langle\hat{S}^{(a)2}\rangle + \langle\hat{S}^{(b)2}\rangle + 2\langle\hat{S}_1^{(a)}\hat{S}_1^{(b)} + \hat{S}_2^{(a)}\hat{S}_2^{(b)}\rangle + 2\langle\hat{S}_3^{(a)}\hat{S}_3^{(b)}\rangle \tag{4}$$
$$\langle\hat{S}^2\rangle = \hbar^2\left(s_a(s_a+1) + s_b(s_b+1) - 2s_a s_b\right) + \langle\hat{S}_1^{(a)}\hat{S}_1^{(b)} + \hat{S}_2^{(a)}\hat{S}_2^{(b)}\rangle \tag{5}$$

Since we have eigenstates of the z-component of spin, azimuthal symmetry guarantees that the latter dot product term expectation value vanishes. Thus our final answer is

$$\boxed{\langle\hat{S}^2\rangle = \hbar^2\left[(s_a - s_b)^2 + s_a + s_b\right]} \tag{6}$$

(b) If $s_a = 1, s_b = 1/2$, then the allowed total angular momentum quantum numbers are $j = 1/2, 3/2$. The relation $\hat{S}^2 |j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$ demonstrates that the relevant eigenvalues are

$$\frac{3\hbar^2}{4}, \frac{15\hbar^2}{4} \tag{7}$$

In the ground state $m_a = s_a, m_b = -s_b$, so $m_a = +1, m_b = -1/2$. To compute the probabilities, let us work out the Clebsch–Gordan coefficients for this case. We start with $j = 3/2$ for which there is only one possibility:

$$\left|\frac{3}{2}, +\frac{3}{2}\right\rangle = |1,+1\rangle \otimes \left|\frac{1}{2}, +\frac{1}{2}\right\rangle \tag{8}$$

Applying the lowering operator to both sides yields

$$\sqrt{3}\left|\frac{3}{2}, +\frac{1}{2}\right\rangle = \sqrt{2}\,|1,0\rangle \otimes \left|\frac{1}{2}, +\frac{1}{2}\right\rangle + |1,+1\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \tag{9}$$
$$\left|\frac{3}{2}, +\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\,|1,0\rangle \otimes \left|\frac{1}{2}, +\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\,|1,+1\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \tag{10}$$

This allows us to find the $|1/2, +1/2\rangle$ state, since it must also be a linear combination of those two kets but further has to be orthogonal to $|3/2, +1/2\rangle$. Thus we obtain, up to a phase,

$$\left|\frac{1}{2}, +\frac{1}{2}\right\rangle = -\sqrt{\frac{1}{3}}\,|1,0\rangle \otimes \left|\frac{1}{2}, +\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\,|1,+1\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \tag{11}$$

We now see that our $m_a = +1, m_b = -1/2$ state can be written as

$$|1,+1\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2}, +\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}, +\frac{1}{2}\right\rangle \tag{12}$$

This is the Clebsch–Gordan decomposition of the ground state we were looking for. The probability that $j = {}^1\!/_2$ is simply the ${}^2\!/_3$ coefficient we see above and for $j = {}^3\!/_2$ the ${}^1\!/_3$. To summarize:

$$\boxed{\begin{aligned} P\left(\frac{3\hbar^2}{4}\right) &= \frac{2}{3} \\ P\left(\frac{15\hbar^2}{4}\right) &= \frac{1}{3} \end{aligned}} \qquad \begin{aligned}(13a)\\(13b)\end{aligned}$$

Note that our expectation value is $7\hbar^2/4$, consistent with our formula from Part (a).