(a) Since $f < 0$, the minimum-energy spin configuration will be the one with the greatest value of $S_3^z$. This occurs when we have eigenstates of $\hat{S}_3$, specifically with $m_a = s_a, m_b = -s_b$. We may compute the expectation value as follows:

$$\langle \hat{S}_3^2 \rangle = (\langle \hat{S}^{(a)}_3 \rangle + \langle \hat{S}^{(b)}_3 \rangle)^2$$

$$\langle \hat{S}_3^2 \rangle = \langle \hat{S}^{(a)}_3 \rangle^2 + \langle \hat{S}^{(b)}_3 \rangle^2 + 2\langle \hat{S}^{(a)}_3 \hat{S}^{(b)}_3 \rangle$$

$$\langle \hat{S}_3^2 \rangle = \langle \hat{S}^{(a)}_3 \rangle^2 + \langle \hat{S}^{(b)}_3 \rangle^2 + 2\langle \hat{S}^{(a)}_3 \hat{S}^{(b)}_3 \rangle + 2\langle \hat{S}^{(a)}_3 \hat{S}^{(b)}_3 \rangle$$

$$\langle \hat{S}_3^2 \rangle = \langle \hat{S}^{(a)}_3 \rangle^2 + \langle \hat{S}^{(b)}_3 \rangle^2 + 2\langle \hat{S}^{(a)}_3 \hat{S}^{(b)}_3 \rangle + 2\langle \hat{S}^{(a)}_3 \hat{S}^{(b)}_3 \rangle$$

$$\langle \hat{S}_3^2 \rangle = \langle \hat{S}^{(a)}_3 \rangle^2 + \langle \hat{S}^{(b)}_3 \rangle^2 + 2\langle \hat{S}^{(a)}_3 \hat{S}^{(b)}_3 \rangle + 2\langle \hat{S}^{(a)}_3 \hat{S}^{(b)}_3 \rangle$$

Since we have eigenstates of the $z$-component of spin, azimuthal symmetry guarantees that the latter dot product term expectation value vanishes. Thus our final answer is

$$\langle \hat{S}_3^2 \rangle = \hbar^2 \left[ (s_a - s_b)^2 + s_a + s_b \right]$$

(b) If $s_a = 1, s_b = \frac{1}{2}$, then the allowed total angular momentum quantum numbers are $j = \frac{1}{2}, \frac{3}{2}$. The relation $\langle \hat{S}_3^2 | j, m \rangle = \hbar^2 j(j + 1) | j, m \rangle$ demonstrates that the relevant eigenvalues are

$$\frac{3\hbar^2}{4}, \frac{15\hbar^2}{4}$$

In the ground state $m_a = s_a, m_b = -s_b$, so $m_a = +1, m_b = -\frac{1}{2}$. To compute the probabilities, let us work out the Clebsch–Gordan coefficients for this case. We start with $j = \frac{3}{2}$ for which there is only one possibility:

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = |1, +1 \rangle \otimes \left| \frac{1}{2}, +\frac{1}{2} \right\rangle$$

Applying the lowering operator to both sides yields

$$\sqrt{3} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \sqrt{2} |1, 0 \rangle \otimes \left| \frac{1}{2}, +\frac{1}{2} \right\rangle + |1, +1 \rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

This allows us to find the $|1/2, +1/2 \rangle$ state, since it must also be a linear combination of those two kets but further has to be orthogonal to $|3/2, +1/2 \rangle$. Thus we obtain, up to a phase,

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = -\sqrt{3} |1, 0 \rangle \otimes \left| \frac{1}{2}, +\frac{1}{2} \right\rangle + \sqrt{2} |1, +1 \rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle$$

We now see that our $m_a = +1, m_b = -1/2$ state can be written as

$$|1, +1 \rangle \otimes \left| \frac{1}{2}, -\frac{1}{2} \right\rangle = \sqrt{\frac{3}{2}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \sqrt{\frac{1}{2}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
This is the Clebsch–Gordan decomposition of the ground state we were looking for. The probability that $j = \frac{1}{2}$ is simply the $2/3$ coefficient we see above and for $j = \frac{3}{2}$ the $1/3$. To summarize:

\[
\begin{array}{c}
P\left(\frac{3h^2}{4}\right) = \frac{2}{3} \\
P\left(\frac{15h^2}{4}\right) = \frac{1}{3}
\end{array}
\] (13a) (13b)

Note that our expectation value is $7h^2/4$, consistent with our formula from Part (a).