Problem M98Q.1

(a) Let $|\Psi\rangle$ denote the (unique) ground state of $H_{\text{spatial}} := K + V(r_a, r_b)$. Then the state

$$|g\rangle := |\Psi\rangle \otimes |m_{s_a} = s_a\rangle \otimes |m_{s_b} = -s_b\rangle$$

is the unique minimizer of the functional

$$|\psi\rangle \mapsto \langle\psi|H|\psi\rangle\,,$$

and by the variational principle must be the ground state of H. It remains to evaluate

$$\langle g|S^2|g\rangle = \underbrace{\langle g|(S^{(a)})^2|g\rangle}_{=s_a\,(s_a+1)} + \underbrace{\langle g|(S^{(b)})^2|g\rangle}_{=s_b\,(s_b+1)} + 2\,\langle g|S^{(a)}\cdot S^{(b)}|g\rangle\,.$$

Since the spins are uncoupled in the ground state, the last term expands to

$$\langle m_{s_a} = s_a|S^{(a)}|m_{s_a} = s_a\rangle \cdot \langle m_{s_b} = s_b|S^{(b)}|m_{s_b} = s_b\rangle = (0,0,s_a)\cdot(0,0,-s_b) = -s_a s_b.$$

Combining the above, we have

$$\langle g|S^2|g\rangle = \boxed{(s_a - s_b)^2 + s_a + s_b.}$$

(b) In this case, the spin component of the wavefunction is

$$|1,1\rangle_a \otimes |1/2,-1/2\rangle_b = \sqrt{\frac{1}{3}}\,|3/2,1/2\rangle_{\text{tot}} + \sqrt{\frac{2}{3}}\,|1/2,1/2\rangle_{\text{tot}}\,,$$

where we have inserted the appropriate Clebsch-Gordan coefficients. Thus a measurement of S^2 in the ground state will result in $15/4$ or $3/4$, with probabilities $1/3$ and $2/3$ respectively.

This yields an expectation value of $7/4$, which is consistent with our formula from (a).

Time: 17 m 56 s