M98M.3 — Fluid Dynamics

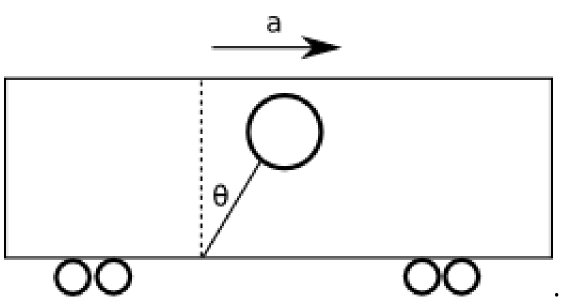
a) Energy must be conserved, so $T = V$ at the hole in the container. Potential energy density is $\frac{V}{volume} = \rho g h$ and kinetic energy density is $\frac{T}{volume} = \frac{1}{2}\rho v^2$. Then:

$$\frac{V}{volume} = \frac{V}{volume}$$

$$\rho g h = \frac{1}{2}\rho v^2$$

$$v = \sqrt{2gh} = \sqrt{2 \cdot 9.8\ m/s \cdot 1\ m} = 4.43\ m/s$$

b) The situation is given by the following image:



The balloon experiences both gravitational acceleration and the acceleration of the train. It moves forward relative to its attachment point since the density of helium is less than that of air. The tension in the string (assisted in the y-direction by gravity) balances the upward and forward forces on the balloon due to air being displaced by helium. Then:

$$T_y = -V\rho_{\mathrm{He}}\, g + V\rho_{\mathrm{air}}\, g$$

$$T_x = V\rho_{\mathrm{air}}\, a$$

$$tan(\theta) = \frac{T_x}{T_y} = \frac{-V\rho_{\text{He}}g + V\rho_{\text{air}}g}{V\rho\text{air}a} = \frac{a}{g}\left(\frac{\rho_{\text{He}}}{\rho_{\text{air}} - \rho_{\text{He}}}\right)$$

$$\theta = tan^{-1}\left(\frac{a}{g}\left(\frac{\rho_{\text{He}}}{\rho_{\text{air}} - \rho_{\text{He}}}\right)\right)$$

This makes sense, as $\theta \to 0$ as $a \to 0$, and becomes $\frac{pi}{2}$ when $\rho_{\text{He}} \to \rho_{\text{air}}$ (at which point the tension is also zero, and the balloon is lying on the floor of the train car).

c) The key realization for this problem is that the equilibrium position of the water in the cylinders (i.e. with no force applied) is the best place from which to measure the displacement of the water. Accordingly, let h_1 be the displacement of the water from equilibrium in the large cylinder, and h_2 be the displacement of the water from equilibrium in the small cylinder. Then the total difference in height between the water in the two cylinders is $h = h_1 + h_2$. Since the water is assumed to be incompressible, the volume change in one cylinder should equal the volume change the other, so $\pi R^2 h_1 = \pi r^2 h_2$. Additionally, by the same token (and the fact that there are no other openings in the system), the pressure p_1 in the large cylinder must be equal to the pressure p_2 in the small cylinder at the interface between them. Then:

$$p_1 = \frac{Mg}{\pi R^2}\ , p_2 = \rho g h_2 \Rightarrow \frac{Mg}{\pi R^2} = \rho g h_2 \Rightarrow h_2 = \frac{M}{\rho \pi R^2}$$

$$h_1 = \left(\frac{r}{R}\right)^2 h_2 = \left(\frac{r}{R}\right)^2 \frac{M}{\rho \pi R^2}$$

$$h = h_1 + h_2 = \frac{M}{\rho \pi R^2} + \left(\frac{r}{R}\right)^2 \frac{M}{\rho \pi R^2} = \frac{M}{\rho \pi R^2}\left(1 + \left(\frac{r}{R}\right)^2\right)$$

Plugging in $R = 1\ \text{m}$, $r = 0.01\ \text{m}$, $M = 100\ \text{kg}$, and $\rho = 1\ \text{g} \cdot \text{cm}^{-3} = 1000\ \text{kg} \cdot \text{m}^{-3}$,

$$h = \frac{100\ \text{kg}}{(1000\ \text{kg}\cdot\text{m}^{-3})\cdot\pi\cdot(1\ \text{m})^2}\left(1 + \left(\frac{0.01\ \text{m}}{1\ \text{m}}\right)^2\right) = 0.0318\ \text{m}$$

d) Directly, this can be done with the Young-Laplace equation: $\Delta p = \gamma\left(\frac{1}{R_x} + \frac{1}{R_y}\right)$, where R_x and R_y are the radii of curvature for the surface under consideration. In this case, the surface here is the surface of a sphere, so $R_x = R_y = R$. Then:

$$\Delta p = \gamma\left(\frac{2}{R}\right)$$

$$\Delta p = (0.05\ \text{N} \cdot \text{m}^{-1})\left(\frac{2}{0.01\ \text{m}}\right) = 10\ \text{N} \cdot \text{m}^{-2}$$

e) Since the flow is laminar and the fluid is incompressible, the volume of fluid in the junction must be conserved. Also, the pressure in the junction must be the average of the pressures at the inlets to the junction. Since $L >> R$, the Darcy–Weisbach equation is not applicable (calculating ΔP along the tube would yield a practically infinite value). Therefore, we can assume that the pressures at the inlets to the junction are the same as at the remote ends of the tubes. In order for fluid to start to flow outward into the 4th tube from the junction, the pressure in the junction must be equal to P', and the velocity must be $v = 0$ in the 4th tube. Then:

$$P_{junct} = P' = (\alpha P' + \alpha P' + 0 + P')/4 \Rightarrow \alpha = \frac{3}{2}$$

Alternately, since the Renolds number is given by $\mathrm{Re} = \frac{\rho v L}{\eta}$, where L is a characteristic length (here equal to the length of the pipe arms), and the Darcy factor $f_D = 64/\mathrm{Re}$, the Darcy-Weisbach equation takes the form $\Delta P = f_D \frac{L}{D} \frac{\rho v^2}{2} = 32\eta \frac{v}{D}$, with $D = 2R$ and $v \equiv \mathrm{average\ velocity}$. Then ΔP is the pressure drop experienced in the fluid after traversing a pipe of length L. Then the pressures at the inlets to the junction are given by (clockwise from the right arm): $\alpha P' - \Delta P$, ΔP, $\alpha P' - \Delta P$, and $P' - \Delta P$. In order to have fluid flowing through the 4th pipe, the pressure in the junction (which is still the average of the pressures at the inlets) must be $P' - \Delta P$. Then:

$$P_{junct} = P' - \Delta P = ((\alpha P' - \Delta P) + (\alpha P' - \Delta P) + \Delta P + (P' - \Delta P))/4$$

$$\Rightarrow \alpha = \frac{3P' - 2\Delta P}{2P'} = \frac{3}{2} + \frac{8\eta v}{RP'}$$

But since $v = 0$ right as the flow is reversing in the 4th pipe, this reduces to $\alpha = \frac{3}{2}$, as above.

One thought on "M98M.3 — Fluid Dynamics"

October 27, 2013 at 2:35 am

a) OK
b) The answer is wrong. It's clear that the direction should be just opposite to the effective gravity in the moving train, while it's not what is given by your formula.
c) You overcomplicted this question. The level difference creates a pressure which balances the extra weight

of the piston and the individual. That's all that you need to care about to get a correct answer.

d) Don't forget that the soap film has two surfaces. Thus an extra factor of 2 is needed.

e) You get a correct answer but your solution doesn't look correct. Actually, I don't understand it, please try to be more clear. In particular your formular for P_{junct} doesn't look OK for me. But maybe I don't understand what you mean.