M98M.2

To begin, let us find the Lagrangian. Taking the horizontal particle at zero potential, the angle θ to be the angle of the constrained particle to the bottom on the ring, and x to be the position of the horizontal particle away from the symmetric center of the system, the kinetic energy of this system is

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{\theta}^2 R^2 \right) \tag{1}$$

while the potential is

$$\begin{aligned} U &= -mgR\cos\theta + \frac{k}{2}\left[(x - R\sin\theta)^2 + (2R - R\cos\theta)^2 \right] && (2) \\ &= -mgR\cos\theta + \frac{k}{2}\left(x^2 - 2Rx\sin\theta + R^2\sin^2\theta + 4R^2 - 4R^2\cos\theta + R^2\cos^2\theta \right) && (3) \\ &= -mgR\cos\theta + \frac{k}{2}\left(x^2 - 2Rx\sin\theta + 5R^2 - 4R^2\cos\theta \right) && (4) \end{aligned}$$

resulting in the Lagrangian

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{\theta}^2 R^2 \right) + mgR\cos\theta - \frac{k}{2}\left(x^2 - 2Rx\sin\theta + 5R^2 - 4R^2\cos\theta \right). \tag{5}$$

We can find the equations of motion, using the Euler-Lagrange equation, resulting in

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + kx - kR\sin\theta = 0 \tag{6}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m\ddot{\theta}R^2 + mgR\sin\theta - kRx\cos\theta + 2kR^2\sin\theta = 0 \tag{7}$$

To find the equilibrium positions, solve at $\partial_q U = 0$, resulting in

$$x = R\sin\theta \tag{8}$$

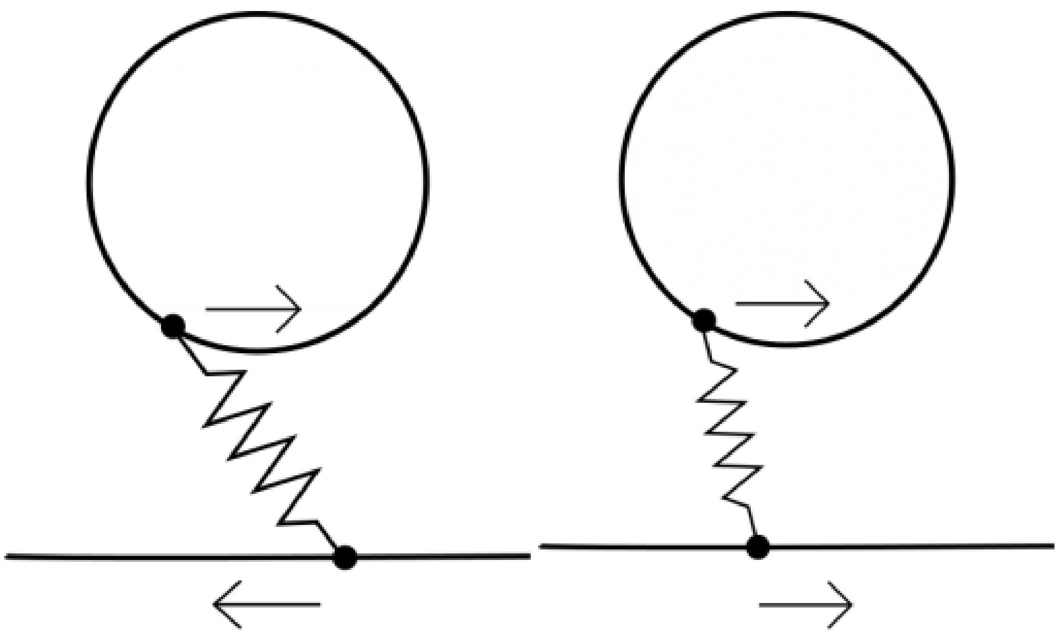
$$mg\sin\theta - kx(\cos\theta + 1) = 0 \tag{9}$$

which is satisfied when $x = 0, \theta = 0,\ \pi$. The second derivatives are

$$\partial_x^2 L = 1 \tag{10}$$

$$\partial_\theta^2 L = mg\cos\theta + kx\sin\theta \tag{11}$$

When $\theta = 0$, we have positive curvature and a stable equilibrium. Otherwise, the equilibrium is unstable. The stable equilibrium point has two normal modes. In one mode, the masses move with each other, while the other mode the masses move in opposite direction. The modes look like the following:



For small oscillattions, we expand the trigonometric functions up to $O(\theta^2)$, neglecting terms of order unity, giving

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{\theta}^2 R^2\right) - \frac{1}{2}mgR\theta^2 - \frac{k}{2}\left[x^2 - 2Rx\theta + 5R^2 + 2R^2\theta^2\right]. \tag{12}$$

Changing the θ coordinate to $R\theta$, this can be represented in the usual matrix notation as

$$T_{ij} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix},\ U_{ij} = \begin{bmatrix} k & -k \\ -k & \Omega \end{bmatrix} \tag{13}$$

where $\Omega = 2k + mgr^{-1}$. To find the normal mode frequencies, we must find the Eigenvalues of the matrix $\omega^2 T - U = 0$, i.e. find the roots of the characteristic equation

$$(k-\lambda)(\Omega-\lambda) - k^2 = k\Omega - k\lambda - \Omega\lambda + \lambda^2 - k^2 = 0 \quad (14)$$

or

$$\lambda^2 - (k+\Omega)\lambda - k^2 + k\Omega = 0 \quad (15)$$

where $\lambda = \omega^2/m$. The roots of this equation are

$$2\lambda = k + \Omega \pm \sqrt{(k+\Omega)^2 - 4(k\Omega - k^2)} \quad (16)$$

The large frequency mode (plus sign) corresponds to the mode where the masses move in opposite directions, since we expect the average spring force to be larger in that case.

One thought on “M98M.2”

October 8, 2013 at 4:46 pm

Good.
There is a computational mistake -- coefficient 2 in (6). Please fix it and all its implications.
Also, in the normal mode when the two masses move with each other, the mass on the line should be a little ahead of the mass on the circle. You can realize that otherwise the configuration doesn't make sense for oscillations.