M98M.2

Solution

Part (a)

Let the angular position of particle P_1 about the center be θ. Let x be the position of the particle P_2 in the x-direction from the center of the circle. Length of the the spring can be written in terms of θ and x using the following relation

$$r^2 = (2R - R\cos\theta)^2 + (x - 2R\cos\theta)^2 \tag{1}$$

The lagrangian of the system is

$$L = T - V = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 \tag{2}$$
$$- [mg(2R - R\cos\theta) + \frac{1}{2}k[(2R - R\cos\theta)^2 + (x - R\sin\theta)^2]]$$

Solving for equations of motion we get

$$\begin{aligned} mR^2\ddot{\theta} &= -mgR\sin\theta - \frac{k}{2}[4R^2\sin\theta - 2xR\cos\theta] \\ &= -(mgR + 2kR^2)\sin\theta + kxR\cos\theta \end{aligned}$$

$$m\ddot{x} = -kx + kR\sin\theta$$

The system is in equilibrium when $\frac{\partial U}{\partial r_i} = 0$ for each co-ordinate i which also means that all forces acting on the system are balanced. So from the equations of motion we have

$$x = R\sin\theta \tag{3}$$

$$(mgR + 2kR^2)\sin\theta = k(R\sin\theta)R\cos\theta \tag{4}$$

Now from the above equation we have either $\sin\theta = 0$ or $\cos\theta = 2 + \frac{mg}{kR}$. The second case is not possible. So we are left with only one solution for equilibrium which is $\sin\theta = 0$ which means $\theta = 0$ or π. \\

As the potential in the problem is continous function, the condition for stable or unstable equilibrium is obtained by looking for minima $\frac{\partial^2 U}{\partial r_i^2} \geq 0$ or maxima $\frac{\partial^2 U}{\partial r_i^2} \leq 0$. \\

$$\frac{\partial^2 U}{\partial \theta^2} = -\frac{\partial F}{\partial \theta} = (mgR + 2kR^2)\cos\theta + kxR\sin\theta \tag{5}$$

$$\frac{\partial^2 U}{\partial x^2} = k \tag{6}$$

For $\theta = 0$ we can see from the above equations that both $\frac{\partial^2 U}{\partial \theta^2}$ and $\frac{\partial^2 U}{\partial x^2}$ are positive and hence its a stable equilibrium. \\

For $\theta = \pi$, we have $\frac{\partial^2 U}{\partial \theta^2} = -(mgR + 2kR^2)$ which is negative and hence the particle P_1 is in unstable equilibrium which mean a slight deviation in θ will decrese the potential energy and it will move away from the equilibrium. On the other hand, $\frac{\partial^2 U}{\partial x^2} = k \geq 0$ so particle P_2 is in stable equilibrium. \\

Part (b) \\

To get the normal modes of oscillation about the equilibrium positions we will consider small excursions in x and θ about $(0,0)$. So we get the following coupled equations

$$mR^2\ddot{\theta} = -(mgR + 2kR^2)\theta + kRx \tag{7}$$

$$m\ddot{x} = -kx + kR\theta \tag{8}$$

Using the test solution $x(t) = A_1\sin(\omega(t)$ and $\theta(t)) = A_2\sin(\omega(t))$, we get following matrix equation

$$-\omega^2\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} \frac{-k}{m} & \frac{kR}{m} \\ \frac{k}{mR} & \frac{-(mg+2kR)}{mR} \end{bmatrix} \times \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} \tag{9}$$

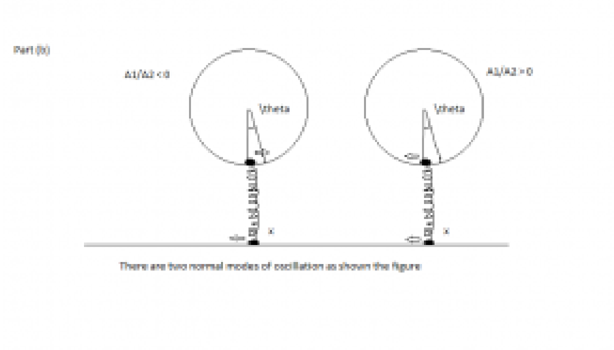
Both of them have to be in phase or out of phase i.e. relative phase should be zero or π. This can be checked easily by putting $x = 0$ in second equation and requiring $\theta = 0$. Thus there are 2 situations possible depending on the sign of $\frac{A_1}{A_2}$ which are shown in the attched figure. The amplitudes can be obtained by obtaining the eigenvectors of the above matrix.\\

Part (c) \\

Solving for eigenavalues we get

$$\omega^2 = \frac{(mg + 3kR) \pm \sqrt{m^2g^2 + 2mgkR + 5k^2R^2}}{2mR} \tag{10}$$

So there are 2 normal modes of oscillation in stable equilibrium and the motion of particles is sinusoidal with frequencies ω_1 and ω_2 .



M98 M.2 solution part B

One thought on “M98M.2”

October 8, 2013 at 5:18 pm

Your solution is correct.
It would make sense to remove all the uncompiled LaTeX garbage.
Originally you had a problem with LaTeX due to empty lines inside of equations (I know, it shouldn't be like that, but that's how WordPress works 🙁