M98M.2

Solution to M98M.2 — Mass Connected by a Spring

a)

Generalized coordinates: x is the distance between the origin and position of P_2. θ is the angle between y axis and the line passes through the center of the circle and position of P_1.

The kinetic and potential energies are:

$$T = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 \tag{1}$$

$$U = mgR(2 - cos\theta) + \frac{1}{2}k[(2R - Rcos\theta)^2 + (x - Rsin\theta)^2] \tag{2}$$

In the equilibrium positions we will have $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \theta} = 0$.

$$\frac{\partial U}{\partial x} = k(x - Rsin\theta) = 0 \tag{3}$$

$$\frac{\partial U}{\partial \theta} = (mgR + 2kR^2)sin\theta - kR^2 sin\theta cos\theta = 0 \tag{4}$$

The two equations above give us either $x = 0, sin\theta = 0$ or $cos\theta = 2 + \frac{mg}{kR}$. Obviously the second solution is impossible. So the equilibria are given by $x = 0, sin\theta = 0$. That means P_2 is at the origin and P_1 is either at the top or at the bottom of the circle.

Now, let us discuss whether they are stable or not. A stable equilibrium requires $\frac{\partial^2 U}{\partial {x_i}^2} > 0$.

$$\frac{\partial^2 U}{\partial x^2} = k \tag{5}$$

$$\frac{\partial^2 U}{\partial \theta^2} = mgRcos\theta + 2kR^2 cos\theta + kRxsin\theta \tag{6}$$

So when $x = 0, \theta = 0$, we get a stable equilibrium and when $x = 0, \theta = \pi$ we get an unstable equilibrium.

b)

The Euler Lagrangian Equations give us

$$\frac{\partial U}{\partial x} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{x}}) = 0 \tag{7}$$

$$\frac{\partial U}{\partial \theta} - \frac{d}{dt}(\frac{\partial T}{\partial \dot{\theta}}) = 0 \tag{8}$$

Consequently

$$k(Rsin\theta - x) = m\ddot{x} \tag{9}$$

$$kRxcos\theta - (mgR + 2kR^2)sin\theta = mR^2\ddot{\theta} \tag{10}$$

Since the oscillation is small, we let $sin\theta = \theta$ and $cos\theta = 1$.

$$k(R\theta - x) = m\ddot{x} \tag{11}$$

$$kRx - (mgR + 2kR^2)\theta = mR^2\ddot{\theta} \tag{12}$$

We suppose the two masses oscillate harmonically. So let $x = A_1 e^{i\omega t}, \theta = A_2\ e^{i\omega t}$. The phase terms are absorbed in A_1 and A_2, so they can be imaginary. After plugging them into the equations of motion and cancelling the common terms, I got the following relations.

$$\begin{bmatrix} \frac{k}{m} & -\frac{kR}{m} \\ -\frac{k}{mR} & \frac{mg+2kR}{mR} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \omega^2 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{13}$$

From the relation above, it is clear that the ratio $\frac{A_1}{A_2}$ is a real number. This means the $x(t)$ and $\theta(t)$ are either in phase or out of phase. Both of them are the normal modes. Since we have 2 dimensions of freedom, we should expect 2 normal modes. When the motions are in phase, P_1 and P_2 are always on the same side of y axis. When they are out of phase, P_1 and P_2 are always on the different sides of y axis.

c)

The square of the frequencies are eigenvalues of the above matrix.

$$\omega^2 = \frac{mg + 3kR \pm \sqrt{m^2g^2 + 5k^2R^2 + 2mgkR}}{2mR} \tag{14}$$

The large frequency is corresponding to the out of phase normal mode, and the small frequency is corresponding to the in phase mode.

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OK, everything looks correct.
