M98M.1

The rope is assumed massless.

Set up polar coordinates with origin at cylinder axis, and $\theta = 0$ pointing right. At any instant, the rope is in contact with the cylinder in the interval $\theta = [0, \pi/2]$, and the force acting on each infinitesimal segment of the rope is in equilibrium.

Let $T(\theta)$ represent the tension in the rope at θ within $\theta = [0, \pi/2]$. Since the mass does not accelerate, $T(0) = mg$. Also, $T(\pi/2) = T$. These are the boundary conditions.

Consider an infinitesimal segment of rope between θ and $\theta + \Delta\theta$. Four forces act on this segment: Tension in the rope on both sides $\overrightarrow{T_R}$ and $\overrightarrow{T_L}$, normal force $\vec{N}$, and friction $\vec{F}$. The tensions are given by:

$$\overrightarrow{T_R} = (-\hat{\theta})T(\theta) \tag{1}$$
$$\overrightarrow{T_L} = \left(\hat{\theta}\cos(\Delta\theta) - \hat{r}\sin(\Delta\theta)\right)T(\theta + \Delta\theta) \tag{2}$$

Equilibrium in $\hat{\theta}$ and $\hat{r}$ directions give:

$$\vec{N} = \hat{r}\sin(\Delta\theta)T(\theta + \Delta\theta) \tag{3}$$
$$\vec{F} = \hat{\theta}(\cos(\Delta\theta)T(\theta + \Delta\theta) - T(\theta)) \tag{4}$$

Also, $\vec{N}$ and $\vec{F}$ are related by:

$$F = \mu N = \mu\sin(\Delta\theta)T(\theta + \Delta\theta) \tag{5}$$

Equate magnitudes of 4 and 5, and divide both sides by $\Delta\theta$:

$$\frac{\cos(\Delta\theta)T(\theta+\Delta\theta)-T(\theta)}{\Delta\theta}=\frac{\sin(\Delta\theta)}{\Delta\theta}\mu T(\theta+\Delta\theta) \tag{6}$$

In the limit $\Delta\theta \to 0$:

$$\frac{dT(\theta)}{d\theta}=\mu T(\theta) \tag{7}$$

Therefore, $T(\theta)=Ce^{\mu\theta}$. Applying boundary condition at $\theta=0$:

$$T(\theta)=mge^{\mu\theta} \tag{8}$$

i.e.

$$T=T(\pi/2)=mge^{\mu\pi/2} \tag{9}$$

One thought on “M98M.1”

December 3, 2013 at 4:26 am

Everything looks correct.