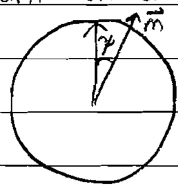
May 1998 #3 (EM)

Earth ($R \sim 6.3$ km) $B = 0.5$ G at equator



magnetic dipole: $\vec{A} = \dfrac{\vec{m}\times\vec{r}}{r^3}$

$$\vec{B} = \nabla\times\vec{A} = \frac{3(\vec{r}\cdot\vec{m})\vec{r} - \vec{m}r^2}{r^5} = \frac{1}{r^3}\left(3\hat{r}\cdot\vec{m}\,\hat{r} - \vec{m}\right) = \frac{m}{r^3}\left(3\hat{r}\cdot\hat{m}\,\hat{r} - \hat{m}\right)$$

$$m \sim BR^3 \cdot \frac{1}{|3\hat{r}\cdot\hat{m}\,\hat{r} - \hat{m}|}$$

Let $\hat{m} = \sin\gamma\,\hat{x} + \cos\gamma\,\hat{z}$

On the equator, $\hat{r} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$

Estimate m by assuming the value for B occurred $\sim \phi = \frac{\pi}{2}$

In this case $\hat{m}\cdot\hat{r} = 0$ (other assumptions change m by $\sim 5\%$)

a. $m = BR^3 = 1.3\cdot 10^{26}$ cm^3 G

b. Radiated power: precession in $\vec{m}$ due to Earth's rotation

$\vec{m} = m[\cos\gamma\,\hat{z} + \sin\gamma(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})]$ $\omega t = \phi$ $\omega = \frac{2\pi}{T}$, $T = 1$ day

z component is static; non-radiating; ignore it

$$\vec{m} = m\sin\gamma\left(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y}\right)$$
$$= m\sin\gamma\,\mathrm{Re}\left[(\hat{x}+i\hat{y})e^{-i\omega t}\right]$$

• Everything carries through using $\tilde{m} = m\sin\gamma(\hat{x} + i\hat{y})$

Radiation zone: $\vec{A} = ik\,\hat{n}\times\vec{m}\,\dfrac{e^{ikr}}{r}$ $k = \frac{\omega}{c}$

$\vec{B} = \nabla\times\vec{A}$ [only the ∇e^{ikr} term stays; $\nabla\frac{1}{r} \sim \frac{1}{r^2}$] $\nabla\hat{n} \sim \frac{1}{r}$

$ik\dfrac{(\hat{n}\times\vec{m})_j}{r}\epsilon_{ijk}\partial_i e^{ikr}$ $\partial_i e^{ikr} = ik\,\partial_i(r)e^{ikr} = ik\,n_i e^{ikr}$

$$\vec{B} = -k^2\hat{n}\times(\hat{n}\times\vec{m})\frac{e^{ikr}}{r} = k^2(\hat{n}\times\vec{m})\times\hat{n}\,\frac{e^{ikr}}{r}$$

Obtain $\vec{E}$ from Maxwell-Ampere Law with $\vec{J}=0$: $\nabla\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = -\frac{i\omega}{c}\vec{E}$

$$\vec{E} = \frac{i}{k}\nabla\times\vec{B} = -\hat{n}\times\vec{B}$$

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{c}{8\pi}\mathrm{Re}\left\{r^2\,\hat{n}\cdot(\vec{E}\times\vec{B}^*)\right\}$$

Since $\vec{m}$ rotates, the power profile also rotates. The average power profile could be calculated (averaged over 1 day), but that isn't important: we can calculate the total power at any given instant

$$\frac{dP}{d\Omega} = \frac{c}{8\pi} \mathrm{Re}\left\{ r^2 |B|^2 \right\} = \frac{c}{8\pi} \cdot k^4 \cdot \left|(\hat{n}\times\vec{m})\times\hat{n}\right|^2 = \frac{c}{8\pi} k^4 \left|\hat{n}\times\vec{m}\right|^2$$

$$= \frac{\omega^4}{8\pi c^3} \sin^2\theta \, |\vec{m}|^2$$

$\vec{m} = m \sin\gamma (\hat{x} + i\hat{y})$

$|\vec{m}|^2 = m^2 \sin^2\gamma \cdot 2$

$$\frac{dP}{d\Omega} = \frac{\omega^4 m^2 \sin^2\gamma \sin^2\theta}{4\pi c^3}$$

$$\int \sin^2\theta \, d\Omega = \frac{8\pi}{3}$$

$$\Rightarrow P = \frac{2}{3} \frac{\omega^4 m^2}{c^3} \sin^2\gamma$$

$\omega = \frac{2\pi}{86400} \frac{rad}{s}$

in ergs/s. $P = 425$ ergs/s

1 joule = 10^7 ergs

$$\boxed{P \simeq 4.2 \cdot 10^{-5} \text{ W}}$$

c. plasma, $n = 10 \text{ cm}^{-3}$

$\omega^2 = \omega_{pe}^2 + k^2c^2$

$\omega_{pe}^2 = \frac{4\pi n e^2}{m_e^2}$ $\qquad \omega_{pe} \sim 1.8 \cdot 10^5 \frac{rad}{s}$ $\qquad \omega_{pe} >> \omega$

$\Rightarrow$ The wave is **cutoff** and cannot propagate

Decay length? $kc = i\sqrt{\omega_{pe}^2 - \omega^2}$

$k \simeq i \frac{\omega_{pe}}{c}$ $\qquad e^{ikx} \sim e^{-x\frac{\omega_{pe}}{c}}$

decay length $\delta = \frac{c}{\omega_{pe}} \sim 1.7$ km