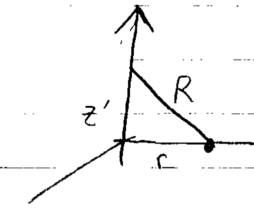
May 1998 #1 (EM)

$I(t) = \begin{cases} 0 & t \le 0 \\ \alpha t & t > 0 \end{cases}$ wire along z-axis



Vector potential at $z=0$

$$\vec{A} = \frac{1}{c}\int \frac{\vec{J}(\vec{r}', t-\frac{R}{c})}{R}\, dV' \qquad R = \sqrt{z'^2 + r^2}$$

$$\vec{A} = \frac{1}{c}\hat{z}\int dz' \frac{I(t-\frac{R}{c})}{R} = \alpha \frac{1}{c}\hat{z}\int dz' \frac{H\left(t - \frac{\sqrt{z'^2+r^2}}{c}\right)\left(t - \frac{\sqrt{z'^2+r^2}}{c}\right)}{\sqrt{z'^2+r^2}}$$

$H(\,) = 0$ if $ct < \sqrt{z'^2+r^2}$

$\Rightarrow \vec{A} = 0$ if $r > ct$

If $r < ct$, limits on integral at $z'^2 = c^2t^2 - r^2$

$$\vec{A} = \frac{\alpha\hat{z}}{c}\left[t\int_{-\sqrt{c^2t^2-r^2}}^{\sqrt{c^2t^2-r^2}} dz' \frac{1}{\sqrt{z'^2+r^2}} - \frac{1}{c}\int_{-\sqrt{c^2t^2-r^2}}^{\sqrt{c^2t^2-r^2}} dz'\right]$$

$$A_z = \frac{\alpha}{c}\left[2t\ln\left(z' + \sqrt{r^2+z'^2}\right)\Big|_0^{\sqrt{c^2t^2-r^2}} - \frac{2}{c}z'\Big|_0^{\sqrt{c^2t^2-r^2}}\right] \qquad ct > r$$

$$A_z = \left\{\frac{2\alpha t}{c}\left[\ln\left(\sqrt{c^2t^2-r^2} + ct\right) - \ln r\right] - \frac{2\alpha}{c^2}\sqrt{c^2t^2-r^2}\right\} H(ct-r)$$

$\vec{B} = \nabla\times\vec{A} \qquad B_\theta = -\frac{\partial A_z}{\partial r}$

$$\boxed{B_\theta = \left\{-\frac{2\alpha t}{c}\left[-\frac{1}{r} + \frac{1}{\sqrt{c^2t^2-r^2}+ct}\left(\frac{-r}{\sqrt{c^2t^2-r^2}}\right)\right] + \frac{2\alpha}{c^2}\frac{(-r)}{\sqrt{c^2t^2-r^2}}\right\} H(ct-r)}$$

$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \qquad E_z = -\frac{1}{c}\frac{\partial A_z}{\partial t}$

$$\boxed{E_z = \left\{-\frac{2\alpha}{c^2}\left[\ln\left(\frac{\sqrt{c^2t^2-r^2}+ct}{r}\right)\right] - \frac{2\alpha t}{c^2}\frac{c + \frac{2c^2t}{2\sqrt{c^2t^2-r^2}}}{\sqrt{c^2t^2-r^2}+ct} + \frac{2\alpha}{c^3}\frac{c^2t}{\sqrt{c^2t^2-r^2}}\right\} H(ct-r)}$$

1) $ct >> r$ $\boxed{B_\theta \to \frac{2\alpha t}{cr} \sim \frac{2I}{cr} \quad (\text{static case})}$

$$E_z \to \frac{-2\alpha}{c^2}\left[\ln\left(\frac{2ct}{r}\right)\right] - \frac{2\alpha t}{c^2}\frac{2c}{2ct} + \frac{2\alpha t}{c^2 t}$$

$$\boxed{E_z \to \frac{-2\alpha}{c^2}\ln\left(\frac{2ct}{r}\right)}$$

2) $ct = r + \epsilon$ $\qquad c^2t^2 - r^2 = (ct-r)(ct+r) \approx \epsilon \cdot 2r$

$$B_\theta \to -\frac{2\alpha t}{c}\left[-\frac{1}{r} - \frac{r}{[\sqrt{2r\epsilon}+r+\epsilon][\sqrt{2r\epsilon}]}\right] - \frac{2\alpha r}{c^2\sqrt{2r\epsilon}}$$

$$\cong -\frac{2\alpha}{c^2}ct\left(-\frac{1}{r} - \frac{r}{r\sqrt{2r\epsilon}+2r\epsilon+\sqrt{2r\epsilon^3}}\right) - \frac{2\alpha}{c^2}\sqrt{\frac{r}{2\epsilon}}$$

$$\cong \frac{2\alpha}{c^2}\left[1 + \frac{\epsilon}{r} + \frac{r+\epsilon}{\sqrt{2r\epsilon}+2\epsilon+\sqrt{\frac{2\epsilon^3}{r}}} - \sqrt{\frac{r}{2\epsilon}}\right]$$

$$\frac{r+\epsilon}{\sqrt{2r\epsilon}+2\epsilon+\sqrt{\frac{2\epsilon^3}{r}}} = \frac{r(1+\frac{\epsilon}{r})}{\sqrt{2r\epsilon}\,(1+\sqrt{\frac{2\epsilon}{r}}+\frac{\epsilon}{r})} \approx \sqrt{\frac{r}{2\epsilon}}\left(1+\frac{\epsilon}{r}\right)\left(1-\sqrt{\frac{2\epsilon}{r}}-\frac{\epsilon}{r}+\frac{2\epsilon}{r}+\ldots\right)$$

$$= \sqrt{\frac{r}{2\epsilon}}\left(1-\sqrt{\frac{2\epsilon}{r}}+\frac{\epsilon}{r}+\frac{\epsilon}{r}+\mathcal{O}\left(\frac{\epsilon}{r}\right)^{3/2}\right)$$

$$= \sqrt{\frac{r}{2\epsilon}} - 1 + \sqrt{\frac{2\epsilon}{r}} + \mathcal{O}\left(\frac{\epsilon}{r}\right)$$

$$\boxed{B_\theta \to \frac{2\alpha}{c^2}\sqrt{\frac{2\epsilon}{r}} + \mathcal{O}\left(\frac{\epsilon}{r}\right)}$$

$$E_z \to -\frac{2\alpha}{c^2}\underbrace{\ln\left(\frac{\sqrt{2r\epsilon}+r+\epsilon}{r}\right)}_{\ln(1+\sqrt{\frac{2\epsilon}{r}}) \approx \sqrt{\frac{2\epsilon}{r}}} - \frac{2\alpha}{c^3}ct\underbrace{\left(\frac{c + \frac{c^2t}{\sqrt{2r\epsilon}}}{\sqrt{2r\epsilon}+r+\epsilon}\right)}_{c\cdot\frac{1}{\sqrt{2r\epsilon}}} + \frac{2\alpha}{c^2}\underbrace{\frac{r+\epsilon}{\sqrt{2r\epsilon}}}_{\sqrt{\frac{r}{2\epsilon}}+\sqrt{\frac{\epsilon}{2r}}}$$

$$\boxed{E_z \to -\frac{2\alpha}{c^2}\sqrt{\frac{2\epsilon}{r}} + \mathcal{O}\left(\frac{\epsilon}{r}\right)}$$

for $ct = r + \epsilon$, $|B_\theta| = |E_z|$