M98E.1 - relativistic fields

Solution to M98E.1 — relativistic fields

In general, the potentials are

$$\phi(\vec{r},t) = \frac{\epsilon_0}{4\pi}\int \frac{\rho(\vec{r}',t_r)}{\left|\vec{r}-\vec{r}'\right|}\,\mathrm{d}\vec{r}' \qquad \vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}',t_r)}{\left|\vec{r}-\vec{r}'\right|}\,\mathrm{d}\vec{r}' \tag{1}$$

Where $t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$ is the retarded time. Since $\rho = 0, V = 0$.

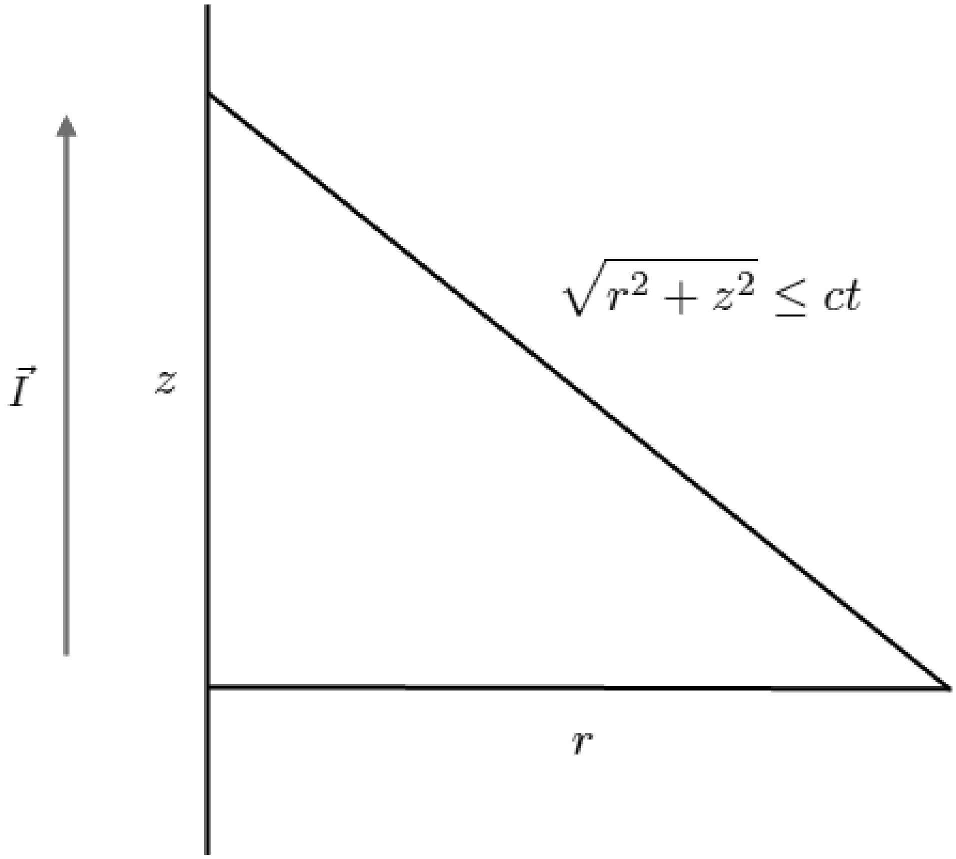

Figure 1: Coordinates used to calculate the vector potential.

$$\vec{A}(\vec{r},t) = \frac{2\mu_0\alpha}{4\pi}\int_0^{z_0}\left[\frac{t}{\sqrt{z^2+r^2}} - \frac{1}{c}\right]\mathrm{d}z \tag{2}$$

$$= \frac{\mu_0\alpha}{2\pi}\left[t\log\left(z+\sqrt{r^2+z^2}\right) - \frac{z}{c}\right]_0^{\sqrt{(ct)^2-r^2}}$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0\alpha}{2\pi}\left[t\log\left(\frac{\sqrt{(ct)^2-r^2}+ct}{r}\right) - \frac{2\sqrt{(ct)^2-r^2}}{c}\right] \tag{3}$$

The above equation is valid for $t \geq \frac{r}{c}$, and $\vec{A} = 0$ for $t < \frac{r}{c}$. Now find the fields.

$$\vec{B} = \nabla\times\vec{A} = -\hat{\phi}\,\frac{\partial A_z}{\partial r} \tag{4}$$

$$\vec{B} = -\frac{\mu_0 \alpha}{2\pi}\left[\frac{rt}{\sqrt{(ct)^2 - r^2} + ct}\left(-\frac{1}{r}\frac{r}{\sqrt{(ct)^2 - r^2}} - \frac{\sqrt{(ct)^2 - r^2} + ct}{r^2}\right) + \frac{2r}{c\sqrt{(ct)^2 - r^2}}\right]\hat{\phi} \tag{5}$$

$$\boxed{\vec{B}_{\text{general}} = \frac{\mu_0 \alpha}{2\pi}\left[\frac{t}{r} + \frac{r}{\sqrt{(ct)^2 - r^2}}\left(\frac{t}{\sqrt{(ct)^2 - r^2} + ct} - \frac{2}{c}\right)\right]\hat{\phi}} \tag{6}$$

$$\vec{E} = -\left(\nabla\phi + \frac{\partial \vec{A}}{\partial t}\right) \tag{7}$$

$$\vec{E} = -\frac{\mu_0 \alpha}{2\pi}\left[\log\left(\frac{\sqrt{(ct)^2 - r^2} + ct}{r}\right) + t\left(\frac{r}{\sqrt{(ct)^2 - r^2} + ct}\right)\frac{1}{r}\left(\frac{c^2 t}{\sqrt{(ct)^2 - r^2}} + c\right) - \frac{2ct}{\sqrt{(ct)^2 - r^2}}\right]\hat{z} \tag{8}$$

$$\vec{E} = -\frac{\mu_0 \alpha}{2\pi}\left[\log\left(\frac{\sqrt{(ct)^2 - r^2} + ct}{r}\right) + \frac{ct}{\sqrt{(ct)^2 - r^2} + ct}\left(\frac{\sqrt{(ct)^2 - r^2} + ct}{\sqrt{(ct)^2 - r^2}}\right) - \frac{2ct}{\sqrt{(ct)^2 - r^2}}\right]\hat{z} \tag{9}$$

$$\boxed{\vec{E}_{\text{general}} = -\frac{\mu_0 \alpha}{2\pi}\left[\log\left(\frac{\sqrt{(ct)^2 - r^2} + ct}{r}\right) - \frac{ct}{\sqrt{(ct)^2 - r^2}}\right]\hat{z}} \tag{10}$$

Equations (6) and (10) are the general magnetic and electric fields. Now we will consider two limits. First, consider $r \ll ct$. Then $\sqrt{(ct)^2 - r^2} = ct\sqrt{1 - \left(\frac{r}{ct}\right)^2} \approx ct$. Thus,

$$\vec{B} \approx \frac{\mu_0 \alpha}{2\pi} \left[\frac{t}{r} + \frac{r}{ct} \left(\frac{t}{2ct} - \frac{2}{c} \right) \right] \hat{\phi} \tag{11}$$

$$\vec{B} = \frac{\mu_0 \alpha t}{2\pi r} \left[1 - \frac{3}{2} \left(\frac{r}{ct} \right)^2 \right] \hat{\phi} \tag{12}$$

$$\boxed{\vec{B}_{\text{non-rel}} \approx \frac{\mu_0 I(t)}{2\pi r} \hat{\phi}} \tag{13}$$

$$\boxed{\vec{E}_{\text{non-rel}} \approx -\frac{\mu_0 \alpha}{2\pi} \left[\log\left(\frac{2ct}{r} \right) - 1 \right] \hat{z}} \tag{14}$$

Second consider the relativistic limit where $r + \epsilon = ct$ for $\epsilon \ll r$. Then $(ct)^2 \approx r^2 + 2r\epsilon$.

$$\vec{B} \approx \frac{\mu_0 I(t)}{2\pi} \left[\frac{1}{r} + \sqrt{\frac{r}{2\epsilon}} \left(\frac{1}{r + \epsilon + \sqrt{2\epsilon r}} - \frac{2}{r + \epsilon} \right) \right] \hat{\phi} \tag{15}$$

$$\vec{B} = \frac{\mu_0 I(t)}{2\pi r} \left[1 + \frac{1}{\sqrt{\frac{2\epsilon}{r}} + \frac{2\epsilon}{r} + \frac{\epsilon}{r}\sqrt{\frac{2\epsilon}{r}}} - \frac{2}{\frac{2\epsilon}{r} + \frac{\epsilon}{r}\sqrt{\frac{2\epsilon}{r}}} \right] \hat{\phi} \tag{16}$$

$$\vec{B} \approx \frac{\mu_0 I(t)}{2\pi r} \left[1 + \frac{1}{\sqrt{\frac{2\epsilon}{r}} + \frac{2\epsilon}{r}} - \frac{2}{\sqrt{\frac{2\epsilon}{r}}} \right] \hat{\phi} \tag{17}$$

$$\boxed{\vec{B}_{\text{ultra-rel}} \approx \frac{\mu_0 I(t)}{2\pi r} \left[1 - \frac{1 + 2\sqrt{\frac{2\epsilon}{r}}}{\sqrt{\frac{2\epsilon}{r}} + \frac{2\epsilon}{r}} \right] \hat{\phi}} \tag{18}$$

$$\vec{E} \approx -\frac{\mu_0 \alpha}{2\pi} \left[\log\left(1 + \sqrt{\frac{2\epsilon}{r}} - \frac{\epsilon}{r} \right) + \frac{r + \epsilon}{\sqrt{2\epsilon r}} \right] \hat{z} \tag{19}$$

$$\vec{E} \approx -\frac{\mu_0 \alpha}{2\pi} \left[\sqrt{\frac{2\epsilon}{r}} + \frac{\epsilon}{r} - \sqrt{\frac{r}{2\epsilon}} - \sqrt{\frac{\epsilon}{2r}} \right] \hat{z} \tag{20}$$

$$\boxed{\vec{E}_{\text{ultra-rel}} \approx \frac{\mu_0 \alpha}{2\pi} \sqrt{\frac{r}{2\epsilon}} \hat{z}} \tag{21}$$

Note that in the above expression we have dropped the $\sqrt{\frac{\epsilon}{r}}$ and $\frac{\epsilon}{r}$ terms since they are smaller than the leading term by at least a factor of $\frac{\epsilon}{r}$.

One thought on "M98E.1 - relativistic fields"

December 11, 2013 at 11:02 pm

You're doing it right.
But it seems to me that in (3) you've got an unwanted factor of 2 before the second term in brackets. And it affected all the later computations.