M98E.1 - relativistic fields

Solution to M98E.1 — relativistic fields

In general, the potentials are

$$\phi(\vec{r}, t) = \frac{\varepsilon_0}{4\pi} \int \frac{\rho(\vec{r}, t_r)}{|\vec{r} - \vec{r}'|} \, d\vec{r}' \quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \, d\vec{r}'$$

(1)

Where $t_r = t - \frac{|\vec{r} - \vec{r}'|}{c}$ is the retarded time. Since $\rho = 0, V = 0$. 
Figure 1: Coordinates used to calculate the vector potential.

The above equation is valid for \( t \geq \frac{r}{c} \), and \( \vec{A} = 0 \) for \( t < \frac{r}{c} \). Now find the fields.

\[
\vec{B} = \nabla \times \vec{A} = -\phi \frac{\partial A_z}{\partial r}
\]
Equations (6) and (10) are the general magnetic and electric fields. Now we will consider two limits. First, consider $r \ll ct$. Then $\sqrt{(ct)^2 - r^2} = ct\sqrt{1 - \left(\frac{r}{ct}\right)^2} \approx ct$. Thus,
Second consider the relativistic limit where \( r + \epsilon = ct \) for \( \epsilon \ll r \). Then \( (ct)^2 \approx r^2 + 2r\epsilon \).

\[
\vec{B} \approx \frac{\mu_0 I(t)}{2\pi} \left[ \frac{1}{r} + \frac{1}{\sqrt{\frac{2\epsilon}{r} + \frac{\epsilon}{r} + \frac{\epsilon}{r}}} - \frac{2}{\sqrt{\frac{2\epsilon}{r} + \frac{\epsilon}{r}}} \right] \hat{\phi} 
\]  

\[
\vec{B} = \frac{\mu_0 \alpha t}{2\pi r} \left[ 1 - \frac{3}{2} \left( \frac{r}{ct} \right)^2 \right] \hat{\phi} 
\]  

\[
\vec{B}_{\text{non-rel}} \approx \frac{\mu_0 I(t)}{2\pi r} \hat{\phi} 
\]  

\[
\vec{E}_{\text{non-rel}} \approx -\frac{\mu_0 \alpha}{2\pi} \left[ \log \left( \frac{2\pi t}{r} \right) - 1 \right] \hat{z} 
\]  

\[
\vec{B} \approx \frac{\mu_0 I(t)}{2\pi r} \left[ 1 + \frac{1}{\sqrt{\frac{2\epsilon}{r} + \frac{\epsilon}{r} + \frac{\epsilon}{r}}} - \frac{2}{\sqrt{\frac{2\epsilon}{r}}} \right] \hat{\phi} 
\]  

\[
\vec{B}_{\text{ultra-rel}} \approx \frac{\mu_0 I(t)}{2\pi r} \left[ 1 - \frac{1 + 2\sqrt{\frac{2\epsilon}{r}}}{\sqrt{\frac{2\epsilon}{r} + \frac{2\epsilon}{r}}} \right] \hat{\phi} 
\]  

\[
\vec{E} \approx -\frac{\mu_0 \alpha}{2\pi} \left[ \log \left( 1 + \sqrt{\frac{2\epsilon}{r} - \frac{\epsilon}{r}} \right) + \frac{r + \epsilon}{\sqrt{2\epsilon r}} \right] \hat{z} 
\]  

\[
\vec{E} \approx -\frac{\mu_0 \alpha}{2\pi} \left[ \sqrt{\frac{2\epsilon}{r}} + \frac{\epsilon}{r} - \sqrt{\frac{r}{2\epsilon}} - \sqrt{\frac{\epsilon}{2r}} \right] \hat{z} 
\]  

\[
\vec{E}_{\text{ultra-rel}} \approx \frac{\mu_0 \alpha}{2\pi} \sqrt{\frac{r}{2\epsilon}} \hat{z} 
\]
Note that in the above expression we have dropped the \( \sqrt{\frac{\varepsilon}{r}} \) and \( \frac{\varepsilon}{r} \) terms since they are smaller than the leading term by at least a factor of \( \frac{\varepsilon}{r} \).

One thought on “M98E.1 - relativistic fields”

December 11, 2013 at 11:02 pm

You're doing it right.
But it seems to me that in (3) you've got an unwanted factor of 2 before the second term in brackets. And it affected all the later computations.