M18T.2 (Brownian Mot^{n})

(a) We are given ē equat^{n} of mot^{n}: $M \dfrac{d^2\vec{r}(t)}{dt^2} + 6\pi\eta b \dfrac{d\vec{r}(t)}{dt} = \vec{F}(t)$

Defin^{g} $\gamma = 6\pi\eta b$ as ē drag coefficient, we have:

$$M\ddot{\vec{r}} + \gamma\dot{\vec{r}} = \vec{F} \implies \ddot{\vec{r}} + \frac{\gamma}{M}\dot{\vec{r}} = \frac{1}{M}\vec{F}$$

Multiply^{g} throughout by $e^{\gamma t/M}$:

$$e^{\gamma t/M}\ddot{\vec{r}} + \frac{\gamma}{M}e^{\gamma t/M}\dot{\vec{r}} = \frac{1}{M}e^{\gamma t/M}\vec{F}$$

$$\implies \frac{d}{dt}\left(e^{\gamma t/M}\dot{\vec{r}}\right) = \frac{1}{M}e^{\gamma t/M}\vec{F}$$

$$\implies e^{\gamma t/M}\dot{\vec{r}} - \vec{v}(0) = \frac{1}{M}\int_0^t dt'\, e^{\gamma t'/M}\vec{F}(t')$$

Thus, we obtain: $\vec{v}(t) = \vec{v}(0)\, e^{-\gamma t/M} + \dfrac{1}{M}\displaystyle\int_0^t dt'\, e^{-\gamma(t-t')/M}\vec{F}(t')$

which makes sense $\because$ it implies ē initial velocity has exponentially decay^{g} impact on $\vec{v}(t)$ while ē subsequent behaviour is solely determined by $\vec{F}(t)$.

(b) To get an express^{n} for $\langle \vec{F}(t)\cdot\vec{F}(t')\rangle$, we can take $\langle v^2(t)\rangle$ & exploit ē equipartit^{n} theorem.

$$\langle v^2(t)\rangle = \langle \vec{v}^{\,2}(0)\, e^{-2\gamma t/M}\rangle + \left\langle \frac{2}{M}e^{-\gamma t/M}\int_0^t dt'\, e^{-\gamma(t-t')/M}\,\vec{v}(0)\cdot\vec{F}(t')\right\rangle$$
$$+ \left\langle \frac{1}{M^2}\int_0^t\int_0^t dt'\,dt''\, e^{-\gamma(t-t')/M}\, e^{-\gamma(t-t'')/M}\,\vec{F}(t')\cdot\vec{F}(t'')\right\rangle$$

Each term individually:

$\langle \vec{v}^{\,2}(0)\, e^{-2\gamma t/M}\rangle = \vec{v}^{\,2}(0)\, e^{-2\gamma t/M}$ (this is a const.)

$\left\langle \dfrac{2}{M}e^{-\gamma t/M}\displaystyle\int_0^t dt'\, e^{-\gamma(t-t')/M}\,\vec{v}(0)\cdot\vec{F}(t')\right\rangle = \dfrac{2}{M}e^{-\gamma t/M}\displaystyle\int_0^t dt'\, e^{-\gamma(t-t')/M}\,\vec{v}(0)\cdot\cancel{\langle\vec{F}(t')\rangle}$ (random force).

$$\left\langle \frac{1}{M^2}\int_0^t\int_0^t dt'\,dt''\, e^{-\gamma(t-t')/M}\, e^{-\gamma(t-t'')/M}\,\vec{F}(t')\cdot\vec{F}(t'')\right\rangle$$
$$= \frac{1}{M^2}\int_0^t\int_0^t dt'\,dt''\, e^{-\gamma(t-t')/M}\, e^{-\gamma(t-t'')/M}\,\langle\vec{F}(t')\cdot\vec{F}(t'')\rangle$$

$$= \frac{1}{M^2}\int_0^t\int_0^t dt'\,dt''\, e^{-\gamma(t-t')/M}\, e^{-\gamma(t-t'')/M}\cdot 3C\,\delta(t'-t'')$$ (ē factor of 3 from ē 3 components of $\vec{F}$).

$$= \frac{3C}{M^2}\int_0^t dt'\, e^{-2\gamma(t-t')/M}$$
$$= \frac{3C}{M^2}e^{-2\gamma t}\int_0^t dt'\, e^{2\gamma t'/M}$$
$$= \frac{3C}{M^2}\cdot\frac{M}{2\gamma}e^{-2\gamma t/M}\left(e^{2\gamma t/M}-1\right)$$
$$= \frac{3C}{M^2}\cdot\frac{M}{2\gamma}\left(1-e^{-2\gamma t/M}\right) = \frac{3C}{2\gamma M}\left(1-e^{-2\gamma t/M}\right)_{//.}$$

$$\Longrightarrow \langle \vec{v}^2(t) \rangle = \vec{v}_0^{\,2} e^{-2\gamma t/M} + \frac{3C}{2\gamma M}\left(1 - e^{-2\gamma t/M}\right)$$

For sufficiently long times, $\bar{c}$ first term dies, & $\bar{c}$ 2^{nd} reduces to $\frac{3C}{2\gamma M}$.

$$\Longrightarrow \langle \vec{v}^2(t) \rangle = \frac{3C}{2\gamma M}$$

However, we also know ∴ in $\bar{c}$ thermodynamic limit, $\frac{1}{2}M\langle v^2 \rangle = \frac{3}{2}k_B T \Longrightarrow \langle v^2 \rangle = \frac{3k_B T}{M}$

Thus, $\frac{3C}{2\gamma M} = \frac{3k_B T}{M} \Longrightarrow C = 2\gamma k_B T$ //.

(c) $\bar{c}$ trick for obtain^g $\bar{c}$ mean squared displacement is to dot $\bar{c}$ equat^n of mot^n w/ $\vec{r}$ & time-average.

$$\left\langle \vec{r}\cdot\ddot{\vec{r}} + \frac{\gamma}{M}\vec{r}\cdot\dot{\vec{r}} \right\rangle = \frac{1}{M}\langle \vec{r}\cdot\vec{F} \rangle$$

$\bar{c}$ RHS is trivially zero, ∵ $\bar{c}$ force is uncorrelated w/ posit^n

$$\vec{r}\cdot\ddot{\vec{r}} = \frac{1}{2}\frac{d^2}{dt^2}(\vec{r}^{\,2}) - \dot{\vec{r}}^{\,2}$$

$$\vec{r}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}(\vec{r}^{\,2})$$

$$\Longrightarrow \left\langle \frac{1}{2}\frac{d^2}{dt^2}(\vec{r}^{\,2}) - \dot{\vec{r}}^{\,2} + \frac{\gamma}{2M}\frac{d}{dt}(\vec{r}^{\,2}) \right\rangle = 0$$

Note ∴ $\left\langle \frac{d^2}{dt^2}(\vec{r}^{\,2}) \right\rangle = \frac{d^2}{dt^2}\langle \vec{r}^{\,2} \rangle$, & similarly for $\frac{d}{dt}$ (∵ $\bar{c}$ average is a sum which commutes w/ $\frac{d}{dt}$).

$$\Longrightarrow \frac{d^2}{dt^2}\langle \vec{r}^{\,2} \rangle + \frac{\gamma}{M}\frac{d}{dt}\langle \vec{r}^{\,2} \rangle = 2\langle \vec{v}^{\,2} \rangle = \frac{6k_B T}{M}$$

Then, $\langle \vec{r}^{\,2} \rangle$ clearly obeys an exponential ansatz, w/ exponents of either 0 (linear time) or $-\frac{\gamma}{M}$ (decay)

Integrat^n: $\langle |\vec{r}(t) - \vec{r}(0)|^2 \rangle = \frac{6k_B T}{M}\cdot\left(\frac{M}{\gamma}\right)^2\left[\frac{\gamma t}{M} - \left(1 - e^{-\gamma t/M}\right)\right]$

For sufficiently long times, $\bar{c}$ graph is linear w/ slope $\frac{6k_B T}{\gamma} = \frac{k_B T}{\pi\eta b}$ (independent of M).

Thus, know^g $\bar{c}$ temperature T, $\bar{c}$ radius b & $\bar{c}$ viscosity η, one could calculate k_B w/ $\bar{c}$ slope read^g.