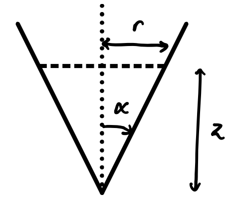
M18M.2 (Particle in a Cone)

(a)



ē particle moves in 3D, w/ a curved planar constraint.

Thus, there must be two generalised coordinates.

Choose ē radial distance r from ē cone axis, & ē angle ϕ around ē axis.

ē particle's posit^{n} is then: $\vec{r} = \langle r\cos\phi,\ r\sin\phi,\ r\cot\alpha \rangle$

$$\Longrightarrow \dot{\vec{r}} = \langle \dot{r}\cos\phi - r\dot{\phi}\sin\phi,\ \dot{r}\sin\phi + r\dot{\phi}\cos\phi,\ \dot{r}\cot\alpha \rangle$$

ē Lagrangian is then: $\mathcal{L} = T - V$

$$\frac{\cos^2}{\sin^2} + \cancel{\sin^2} = \frac{1}{\sin^2}$$

$$= \frac{1}{2}m\dot{\vec{r}}^{\,2} - mgr\cot\alpha$$

$$= \frac{1}{2}m\dot{r}^2\csc^2\alpha + \frac{1}{2}mr^2\dot{\phi}^2 - mgr\cot\alpha$$

ē Euler-Lagrange equat^{n}s are: ϕ equat^{n}: $p_\phi = mr^2\dot{\phi} = \text{const.}$

r equat^{n}: $m\ddot{r}\csc^2\alpha = mr\dot{\phi}^2 - mg\cot\alpha$ //.

(b) For a circular orbit of fixed height, $\dot{r} = \ddot{r} = 0$

$$\Longrightarrow mr_0\dot{\phi}^2 = mg\cot\alpha$$

$$\dot{\phi}^2 = \omega_0^2 = \frac{g}{r_0}\cot\alpha \Longrightarrow \omega_0 = \sqrt{\frac{g}{r_0}\cot\alpha}$$ //.

(c) If ē circular orbit is slightly perturbed, we will have: $r = r_0 + \varepsilon$, w/ $\varepsilon \ll r_0$, & $\ddot{r} = \ddot{\varepsilon}$

ē r equat^{n} then becomes: $m\ddot{r}\csc^2\alpha = \frac{p_\phi^2}{mr^3} - mg\cot\alpha$

$$m\ddot{\varepsilon}\csc^2\alpha = \frac{p_\phi^2}{mr_0^3}\left(1 + \frac{\varepsilon}{r_0}\right)^{-3} - mg\cot\alpha$$

$$m\ddot{\varepsilon}\csc^2\alpha \approx \frac{p_\phi^2}{mr_0^3}\left(1 - \frac{3\varepsilon}{r_0}\right) - mg\cot\alpha$$

$$\Longrightarrow \ddot{\varepsilon} = -\frac{3p_\phi^2}{m^2r_0^4}\varepsilon\sin^2\alpha$$ ($\because$ ē other terms cancel out for ē circular orbit).

Thus, ē freq. of ē small oscillat^{n}s is: $\omega_{osc}^2 = \frac{3p_\phi^2}{m^2r_0^4}\sin^2\alpha$

$$= (3\sin^2\alpha)\,\omega_0^2 \Longrightarrow \Omega = (\sqrt{3}\sin\alpha)\,\omega_0$$

(d) For ē orbit to be closed, we need an integer multiple of small oscillat^{n}s to occur for each orbit of ϕ.

Thus, we require $\Omega = n\omega_0 \Longrightarrow \sin\alpha = \frac{n}{\sqrt{3}}$

Thus, ē orbit is only closed for $n = 1$ ($\alpha = \arcsin\frac{1}{\sqrt{3}}$) //.