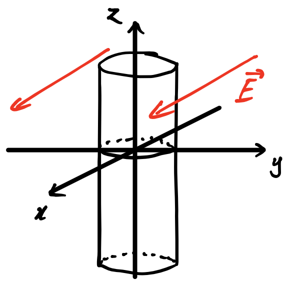
M18E.3 (Dielectric Cylinder in an Electric Field)

(a)



Place ē cylinder's axis along ē z-axis, let $\vec{E}$ be oriented along $\hat{x}$ so $\vec{E} = E_0 \hat{x}$

By symmetry along ē z-axis, ē potential cannot depend on z.

Then, ē general solut^{n} is: (V_{in} for $r < a$, V_{out} for $r > a$)

$$V(r,\theta) = A_0 + B_0 \ln r + \sum_{m=1}^{\infty} \left(A_m r^m + B_m r^{-m}\right)\left(C_m \cos(m\theta) + D_m \sin(m\theta)\right)$$

First, V_{in} must have $B_m = 0$ for all $m \geq 0$, otherwise it diverges at ē z-axis.

$$V_{in}(r,\theta) = A_0 + \sum_{m=1}^{\infty} r^m \left(A_m \cos(m\theta) + B_m \sin(m\theta)\right)$$

$$V_{out}(r,\theta) = F_0 + G_0 \ln r + \sum_{m=1}^{\infty} r^m \left(F_m \cos(m\theta) + G_m \sin(m\theta)\right) + \sum_{m=1}^{\infty} r^{-m}\left(H_m \cos(m\theta) + J_m \sin(m\theta)\right)$$

Now, we impose boundary condit^{n}s:

(1) $\vec{E}$ at $r \to \infty$ must go to $E_0 \hat{x} \Longrightarrow V_{out}(r \to \infty, \theta) \longrightarrow -E_0 r\cos\theta$

$$\Longrightarrow F_1 = -E_0, \; G_0 = F_{m \neq 1} = G_m = 0$$

$$\Longrightarrow V_{out}(r,\theta) = F_0 - E_0 r\cos\theta + \sum_{m=1}^{\infty} r^{-m}\left(H_m \cos(m\theta) + J_m \sin(m\theta)\right)$$

(2) Continuity of V: $V_{in}(a) = V_{out}(a)$

$$\Longrightarrow A_0 + \sum_{m=1}^{\infty} a^m \left(A_m \cos(m\theta) + B_m \sin(m\theta)\right)$$

$$= F_0 - E_0 a\cos\theta + \sum_{m=1}^{\infty} a^{-m}\left(H_m \cos(m\theta) + J_m \sin(m\theta)\right)$$

$$\Longrightarrow A_0 = F_0, \; a^m B_m = a^{-m} J_m, \; a^m A_m = a^{-m} H_m \; (\text{for } m \neq 1) \quad (1)$$

for $m = 1$: $A_1 a\cos\theta = -E_0 a\cos\theta + \frac{H_1}{a}\cos\theta \Longrightarrow A_1 = -E_0 + \frac{H_1}{a^2}$

(2) Continuity of $\vec{D}$: $\varepsilon \left.\frac{\partial V_{in}}{\partial r}\right|_{r=a} = \varepsilon_0 \left.\frac{\partial V_{out}}{\partial r}\right|_{r=a}$

$$\varepsilon \left\{ \left(-E_0 + \frac{H_1}{a^2}\right)\cos\theta + \frac{J_1}{a^2}\sin\theta + \sum_{m=2}^{\infty} m a^{m-1}\left(A_m \cos(m\theta) + B_m \sin(m\theta)\right)\right\}$$

$$= \varepsilon_0 \left\{ -E_0 \cos\theta + \sum_{m=1}^{\infty} (-m) a^{-m-1}\left(H_m \cos(m\theta) + J_m \sin(m\theta)\right)\right\}$$

$$\Longrightarrow (m \neq 1) \; m a^{m-1} A_m = -m a^{-m-1} H_m \xrightarrow{(1)} \frac{m}{a} a^{-m} H_m = -m a^{-m-1} H_m \Longrightarrow H_m = A_m = 0.$$

$$m a^{m-1} B_m = -m a^{-m-1} J_m \xrightarrow{(1)} \frac{m}{a} a^{-m} J_m = -m a^{-m-1} J_m \Longrightarrow J_m = B_m = 0$$

(for $m = 1$): $\varepsilon\left(-E_0 + \frac{H_1}{a^2}\right) = \varepsilon_0\left(-E_0 - \frac{H_1}{a^2}\right)$

$$\Longrightarrow (\varepsilon - \varepsilon_0) E_0 = (\varepsilon + \varepsilon_0)\frac{H_1}{a^2} \Longrightarrow H_1 = a^2 \frac{\varepsilon - \varepsilon_0}{\varepsilon + \varepsilon_0}$$

$\therefore$ we obtain $\bar{c}$ potentials as: (ignore $\bar{c}$ const. term $\because$ potential is defined up to a const.)

$$\begin{cases} V_{in}(r,\theta) = -\dfrac{2\varepsilon_0}{\varepsilon+\varepsilon_0} E_0 r\cos\theta \\ V_{out}(r,\theta) = \left(\dfrac{\varepsilon-\varepsilon_0}{\varepsilon+\varepsilon_0}\dfrac{a^2}{r^2} - 1\right) E_0 r\cos\theta \end{cases} \quad /\!/.$$

(b) $$\vec{E}_{in} = -\nabla V_{in}$$

$$= -\frac{\partial V_{in}}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V_{in}}{\partial\theta}\hat{\theta}$$

$$= \frac{2\varepsilon_0}{\varepsilon+\varepsilon_0} E_0 \cos\theta\,\hat{r} - \frac{2\varepsilon_0}{\varepsilon+\varepsilon_0} E_0 \sin\theta\,\hat{\theta} \equiv \frac{2\varepsilon_0}{\varepsilon+\varepsilon_0} E_0\,\hat{x}$$

$$\vec{D}_{in} = \varepsilon\vec{E}_{in} = \left(\frac{2\varepsilon_0\varepsilon E_0}{\varepsilon+\varepsilon_0}\right)(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \equiv \frac{2\varepsilon_0\varepsilon E_0}{\varepsilon+\varepsilon_0}\hat{x} \quad /\!/.$$

(c) $\bar{c}$ bound surface charge is obtained as $\sigma_b = \vec{P}\cdot\hat{n}$, where $\vec{P}$ is $\bar{c}$ polarisat$^{\underline{o}}$.

$$\vec{P} = \vec{D} - \varepsilon_0\vec{E}$$

$$= \frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0}(\varepsilon-\varepsilon_0)\,\hat{x}$$

$$\sigma_b = \vec{P}\cdot\hat{r}$$

$$= \frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0}(\varepsilon-\varepsilon_0)\cos\theta\,\hat{r} \quad /\!/.$$

$\bar{c}$ vol. bound charge density is: $\rho_b = -\nabla\cdot\vec{P}$

$$= -\frac{1}{r}\frac{\partial(rP_r)}{\partial r} - \frac{1}{r}\frac{\partial P_\theta}{\partial\theta}$$

$$= -\frac{1}{r}\frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0}(\varepsilon-\varepsilon_0)\cos\theta + \frac{1}{r}\frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0}(\varepsilon-\varepsilon_0)\sin\theta$$

$= 0$ (expected $\because$ all charge should reside at $\bar{c}$ surface).

(d) $\bar{c}$ electrostatic energy is simply:

$$U = \frac{1}{2}\int d\tau'\,\vec{D}\cdot\vec{E}$$

$$\frac{U}{l} = \frac{1}{2}\int_0^{2\pi} d\theta \cdot \int_0^a dr \cdot r\left(\frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0}\right)^2 \varepsilon$$

$$= \pi\varepsilon\left(\frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0}\right)^2 \cdot \frac{1}{2}a^2$$

$$= \frac{\pi\varepsilon a^2}{2}\left(\frac{2\varepsilon_0 E_0}{\varepsilon+\varepsilon_0}\right)^2 \quad /\!/.$$