PROBLEM M18E.3

(a) Assume that E_0 is in the $\hat{\mathbf{x}}$ direction. Define the functions

$$P_n(\theta, r, z) := \cos(n\theta) r^n.$$

Then any solution to Laplace's equation with the relevant symmetry must be a linear combination of the P_n. (There is also a $\log r$ solution that turns out not to be necessary.)

Without the cylinder, the electric potential is

$$\Phi(r, \theta, z) = -E_0 r \cos\theta = -E_0 P_1.$$

The $\mathbf{E}$-field inside the dielectric cylinder must be uniform with possibly reduced strength; i.e.

$$\Phi(r, \theta, z) = -(1 - \alpha) E_0 \, P_1$$

for some $0 < \alpha < 1$. Outside the cylinder, to ensure Φ is continuous, we must have

$$\Phi(r, \theta, z) = -E_0 P_1 + \alpha E_0 P_{-1} a^2.$$

The $\mathbf{D}$-field must be continuous along the x-axis. We compute

$$\mathbf{E}_{r \to a^-} = (1 - \alpha) E_0 \, \hat{\mathbf{x}} \quad \text{and} \quad \mathbf{E}_{r \to a^+} = (1 + \alpha) E_0 \, \hat{\mathbf{x}},$$

so imposing $\mathbf{D}_{r \to a^-} = \mathbf{D}_{r \to a^+}$ yields

$$\epsilon \, (1 - \alpha) = \epsilon_0 \, (1 + \alpha),$$

which yields the solution

$$\alpha = \frac{\epsilon - \epsilon_0}{\epsilon + \epsilon_0}.$$

Combining the above, we obtain

$$\Phi(r, \theta, z) = -\frac{2}{1 + \epsilon_r} E_0 r \cos\theta$$

inside the cylinder, and

$$\Phi(r, \theta, z) = -E_0 \cos\theta \left(r + \frac{\epsilon_r - 1}{\epsilon_r + 1} \frac{a^2}{r} \right)$$

outside the cylinder, where $\epsilon_r := \epsilon / \epsilon_0$.

(b) Inside the cylinder, we have

$$\mathbf{E} = \frac{2}{1 + \epsilon_r} E_0 \hat{\mathbf{x}},$$

and so

$$\mathbf{D} = \frac{2\epsilon}{1 + \epsilon_r} E_0 \hat{\mathbf{x}}.$$

(c) Since $\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E}$ is uniform within the cylinder, there is no volume bound charge.

On the surface, we have

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = 2\epsilon_0 \frac{\epsilon_r - 1}{\epsilon_r + 1} E_0 \cos\theta.$$

(d) We compute

$$\frac{U}{L} = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, \mathrm{d}A = \frac{2\pi a^2 \epsilon E_0^2}{(1 + \epsilon_r)^2}.$$