28 Dec 2021

3. **Tritium Beta Decay**

The tritium nucleus is radioactive and decays to He^3 with the emission of an electron and an antineutrino ($T \rightarrow He^3 + e^- + \bar{\nu}$).

(a) Assuming that the electron in the tritium atom is originally in its ground state, what is the probability of finding the electron in the resulting He^3 ion also in its $1s$ ground state immediately after the decay? You can assume that, as far as the electron is concerned, all that happens is that the nucleus suddenly changes its charge from $+1$ to $+2$. The other newly-produced electron is emitted with such high energy that it effectively leaves the atom immediately.

(b) What are the probabilities of finding the electron in the He^3 ion in each of its three $2p$ excited states immediately after the decay?

(c) What is the expectation value of the energy of the atomic electron immediately after the decay (old wave function, new Hamiltonian)?

M17 Q.3

a) $\psi_{gs,Z} = \sqrt{\frac{Z^3}{\pi a^3}} e^{-Zr/a} \leftarrow a = \frac{4\pi\varepsilon_0 \hbar^2}{e^2 m}$

$\psi_T = \psi_{gs,1} = \sqrt{\frac{1}{\pi a^3}} e^{-r/a}$

$\psi_{He} = \psi_{gs,2} = \sqrt{\frac{8}{\pi a^3}} e^{-2r/a}$

$$c = \langle \psi_{He} | \psi_T \rangle = \frac{\sqrt{8}}{\pi a^3}\left(4\pi \int_0^\infty r^2 e^{-3r/a} dr\right) = \frac{\sqrt{8}}{\pi a^3}\left(4\pi \int_0^\infty r^2 e^{-2(\frac{3}{2})r/a} dr\right) = \frac{\sqrt{8}}{\pi a^3} \frac{\pi a^3}{(3/2)^3} \langle \psi_{gs,\frac{3}{2}} | \psi_{gs,\frac{3}{2}} \rangle$$
$$= \left(\frac{\sqrt{8}}{3}\right)^3$$

$$\boxed{P = c^2 = \left(\frac{8}{9}\right)^3}$$

c) $H_{He} = \frac{p^2}{2m} - \frac{2e^2}{4\pi\varepsilon_0 r} \leftarrow Z = 2$

$= H_T - \frac{e^2}{4\pi\varepsilon_0 r}$

$$E = \langle \psi_T | H_T | \psi_T \rangle - \frac{e^2}{4\pi\varepsilon_0} \langle \psi_T | \tfrac{1}{r} | \psi_T \rangle \leftarrow \langle \psi_T | \tfrac{1}{r} | \psi_T \rangle = \frac{1}{a}$$
$$= \frac{-\hbar^2}{2ma^2} - \frac{e^2}{4\pi\varepsilon_0 a} = \frac{-\hbar^2}{2ma^2} - \frac{\hbar^2}{ma^2} = -3\frac{\hbar^2}{2ma^2}$$

$$\boxed{E = -3\frac{\hbar^2}{2ma^2} \approx -40.8\,eV}$$

b) $\psi_{211} = \frac{1}{8\sqrt{\pi a^3}} \frac{r}{a} \sin\theta e^{i\phi} e^{-r/a}$

$\psi_{21-1} = \frac{1}{8\sqrt{\pi a^3}} \frac{r}{a} \sin\theta e^{-i\phi} e^{-r/a}$

$\psi_{2p_x} = \frac{1}{\sqrt{2}}(\psi_{211} + \psi_{21-1}) = \frac{1}{4\sqrt{2\pi a^3}} \frac{r}{a} \sin\theta \cos\phi\, e^{-r/a}$

$\psi_{2p_y} = \frac{-i}{\sqrt{2}}(\psi_{211} - \psi_{21-1}) = \frac{1}{4\sqrt{2\pi a^3}} \frac{r}{a} \sin\theta \sin\phi\, e^{-r/a}$

$\psi_{2p_z} = \psi_{210} = \frac{1}{4\sqrt{2\pi a^3}} \frac{r}{a} \cos\theta\, e^{-r/a}$

Orbitals

$$C_{2p_x} \propto \int_0^{2\pi} \cos\phi \, d\phi = 0 \rightarrow P_{2p_x} = |C_{2p_x}|^2 = 0$$
$$C_{2p_y} \propto \int_0^{2\pi} \sin\phi \, d\phi = 0 \rightarrow P_{2p_y} = |C_{2p_y}|^2 = 0$$
$$C_{2p_z} \propto \int_0^{\pi} \sin\theta \cos\theta \, d\theta = 0 \rightarrow P_{2p_z} = |C_{2p_z}|^2 = 0$$

$m = \pm 1$ eigenstates

$$C_{211} \propto \int_0^{2\pi} e^{i\phi} d\phi = \int_0^{2\pi} \cos\phi \, d\phi + i \int_0^{2\pi} \sin\phi \, d\phi = 0 \rightarrow P_{211} = |C_{211}|^2 = 0$$
$$C_{21-1} \propto \int_0^{2\pi} e^{-i\phi} d\phi = \int_0^{2\pi} \cos\phi \, d\phi - i \int_0^{2\pi} \sin\phi \, d\phi = 0 \rightarrow P_{21-1} = |C_{21-1}|^2 = 0$$

$$\boxed{P_{2p} = 0}$$