01 Jan 2022

Section A. Quantum Mechanics

1. **Atom in a Cavity**

Consider a simple model of a cavity mode of the electromagnetic field interacting with a two-state system representing an atom which can absorb or emit cavity mode photons. The Hamiltonian for this system is taken to be the sum of a harmonic oscillator representing the cavity mode, a term which splits the energy of the two-state system, and an interaction term:

$$H_0 = \epsilon_c a^\dagger a + \frac{1}{2}\epsilon_a \sigma_z + \gamma(a\sigma_+ + a^\dagger \sigma_-) \ ,$$

where $a^\dagger$ is the usual harmonic oscillator creation operator, ϵ_c is the energy of a cavity mode photon, ϵ_a is the energy difference between the ground (g) and excited (e) states of the two-state system, and the σ matrices act on the two-state system spanned by $\{|g\rangle, |e\rangle\}$.

(a) Show that this Hamiltonian can be block diagonalized in two-state subspaces spanned by the states $|g, n+1\rangle, |e, n\rangle$ (where n is the level of the harmonic oscillator, i.e. the number of cavity photons). Diagonalize this Hamiltonian to obtain an expression for the splitting between the two energy eigenstates in each block. Show that this splitting is smallest when the atom and the radiation are in resonance: $\epsilon_a = \epsilon_c$.

For the rest of this problem let $\epsilon_a = \epsilon_c$, so we are at resonance.

(b) Suppose that at time $t = 0$ the system is in the state $|e, n\rangle$. Because this is not an energy eigenstate, the state will evolve in time. Derive expressions for $p_g(t)$ and $p_e(t)$ (the probabilities of finding the atom in its ground and excited states respectively), as a function of time. Show that these probabilities undergo oscillations with a definite period.

(c) Suppose that at $t = 0$ the atom is in the excited state and the cavity is in a superposition of cavity modes: $\sum p_n |n>$ where $p(n) = (1/\sqrt{2\pi n_0}) \exp((n - n_0)^2/2n_0)$. When n_0 is large, this is pretty close to a *coherent state* of excitation of the cavity, which is as close as we get in quantum mechanics to a classical state of the field. Derive formal expressions for $p_e(t)$ and $p_g(t)$. For n_0 large, with what period do these probabilities initially oscillate? On roughly what time scale do these initial oscillations first dephase?

M17 Q.1

Convention: $\sigma_\pm = \sigma_x \pm i\sigma_y \to \sigma_\pm |\mp s\rangle = 2|\pm s\rangle$

a) $H_0|g, n+1\rangle = \varepsilon_c(n+1)|g, n+1\rangle - \frac{1}{2}\varepsilon_a|g, n+1\rangle + 2\gamma\sqrt{n+1}\,|e, n\rangle$

$= [n\varepsilon_c + (\varepsilon_c - \frac{1}{2}\varepsilon_a)]|g, n+1\rangle + 2\gamma\sqrt{n+1}\,|e, n\rangle$

$H_0|e, n\rangle = \varepsilon_c(n)|e, n\rangle + \frac{1}{2}\varepsilon_a|e, n\rangle + 2\gamma\sqrt{n+1}\,|g, n+1\rangle$

$= [n\varepsilon_c + \frac{1}{2}\varepsilon_a]|e, n\rangle + 2\gamma\sqrt{n+1}\,|g, n+1\rangle$

Forms block diagonal of these dressed states. Looking at just one block

$$H_0 = \begin{pmatrix} [n\varepsilon_c + (\varepsilon_c - \frac{1}{2}\varepsilon_a)] & 2\gamma\sqrt{n+1} \\ 2\gamma\sqrt{n+1} & [n\varepsilon_c + \frac{1}{2}\varepsilon_a] \end{pmatrix} \to |H_0 - E_\pm I| = 0 \text{ gives eigen energies}$$

$$([n\varepsilon_c + (\varepsilon_c - \tfrac{1}{2}\varepsilon_a)] - E_\pm)([n\varepsilon_c + \tfrac{1}{2}\varepsilon_a] - E_\pm) - 4\gamma(n+1) = 0$$

$$E_\pm^2 + (-[n\varepsilon_c + (\varepsilon_c - \tfrac{1}{2}\varepsilon_a)] - [n\varepsilon_c + \tfrac{1}{2}\varepsilon_a])E_\pm + ([n\varepsilon_c + (\varepsilon_c - \tfrac{1}{2}\varepsilon_a)][n\varepsilon_c + \tfrac{1}{2}\varepsilon_a] - 4\gamma(n+1)) = 0$$

$$E_\pm^2 + (-2n-1)\varepsilon_c E_\pm + (n^2\varepsilon_c^2 + n\varepsilon_c^2 + \tfrac{1}{2}\varepsilon_a(\varepsilon_c - \tfrac{1}{2}\varepsilon_a) - 4\gamma(n+1)) = 0$$

$$E_\pm = \frac{1}{2}\left[(2n+1)\varepsilon_c \pm \sqrt{4n^2\varepsilon_c^2 + 4n\varepsilon_c^2 + \varepsilon_c^2 - 4n^2\varepsilon_c^2 - 4n\varepsilon_c^2 - 2\varepsilon_a(\varepsilon_c - \tfrac{1}{2}\varepsilon_a) + 16\gamma(n+1)}\right]$$

$$= \frac{1}{2}\left[(2n+1)\varepsilon_c \pm \sqrt{\varepsilon_c^2 - \varepsilon_a(2\varepsilon_c - \varepsilon_a) + 16\gamma(n+1)}\right]$$

$\Delta \equiv E_+ - E_- = \sqrt{\varepsilon_c^2 - \varepsilon_a(2\varepsilon_c - \varepsilon_a) + 16\gamma(n+1)}$

minimize $\Delta \to$ minimize Δ^2 ($\Delta > 0$, so this works)

$\frac{\partial \Delta^2}{\partial \varepsilon_a} = -2\varepsilon_c + \varepsilon_a + \varepsilon_a = 0 \to \varepsilon_c = \varepsilon_a$ at minimum

$$\boxed{\Delta = \sqrt{\varepsilon_c^2 - \varepsilon_a(2\varepsilon_c - \varepsilon_a) + 16\gamma(n+1)}, \text{ minimized at } \varepsilon_a = \varepsilon_c}$$

b) $\varepsilon_a = \varepsilon_c$

$$H_0 = \begin{pmatrix} (n+\frac{1}{2})\varepsilon_c & 2\gamma\sqrt{n+1} \\ 2\gamma\sqrt{n+1} & (n+\frac{1}{2})\varepsilon_c \end{pmatrix} = (n+\tfrac{1}{2})\varepsilon_c I + 2\gamma\sqrt{n+1}\,\sigma_x$$

$$P_e(t) = |\langle e,n|U(t)|e,n\rangle|^2 \leftarrow U(t) = e^{-i\frac{t}{\hbar}H_0} = e^{-it\frac{\varepsilon_c}{\hbar}(n+\frac{1}{2})I}\, e^{-it\frac{\gamma}{\hbar}2\sqrt{n+1}\sigma_x}$$

$$\langle e,n|U(t)|e,n\rangle = \langle e,n|e^{-it\frac{\varepsilon_c}{\hbar}(n+\frac{1}{2})I}|e,n\rangle\langle e,n|e^{-it\frac{\gamma}{\hbar}2\sqrt{n+1}\sigma_x}|e,n\rangle$$

$$= e^{-it\frac{\varepsilon_c}{\hbar}(n+\frac{1}{2})}\langle e,n|e^{i(t\frac{\gamma}{\hbar}2\sqrt{n+1})(-\sigma_x)}|e,n\rangle \leftarrow e^{ia(\hat{n}\cdot\vec{\sigma})} = I\cos(a) + i(\hat{n}\cdot\vec{\sigma})\sin(a)$$

$$= e^{-it\frac{\varepsilon_c}{\hbar}(n+\frac{1}{2})}\langle e,n|I\cos(\omega t) - i\sigma_x\sin(\omega t)|e,n\rangle \leftarrow \omega \equiv 2\frac{\gamma}{\hbar}\sqrt{n+1}$$

$$= e^{-it\frac{\varepsilon_c}{\hbar}(n+\frac{1}{2})}\cos(\omega t)$$

$$\boxed{\begin{aligned} P_e(t) &= \cos^2\left(2\frac{\gamma}{\hbar}\sqrt{n+1}\,t\right) \\ P_g(t) &= 1 - P_e(t) = \sin^2\left(2\frac{\gamma}{\hbar}\sqrt{n+1}\,t\right) \end{aligned}}$$

$\leftarrow$ these do be oscillating, not gonna lie

c) $|\psi_e\rangle = \sum_{n=0}^{\infty} p_n|e,n\rangle$

$$\boxed{\begin{aligned} P_e &= \sum_{n=0}^{\infty} p_n^2\cos^2\left(2\frac{\gamma}{\hbar}\sqrt{n+1}\,t\right) \leftarrow p_n^2 = \frac{1}{2\pi n_0}\exp\left[\frac{(n-n_0)^2}{n_0}\right] \\ &= \frac{1}{2\pi n_0}\sum_{n=0}^{\infty}\exp\left[\frac{(n-n_0)^2}{n_0}\right]\cos^2\left(2\frac{\gamma}{\hbar}\sqrt{n+1}\,t\right) \\ P_g &= \frac{1}{2\pi n_0}\sum_{n=0}^{\infty}\exp\left[\frac{(n-n_0)^2}{n_0}\right]\sin^2\left(2\frac{\gamma}{\hbar}\sqrt{n+1}\,t\right) \end{aligned}}$$

Large $n \rightarrow$ high peak in n_0 ($p_{n_0} \simeq 1$)

$$\omega = 2\frac{\gamma}{\hbar}\sqrt{n_0+1} \approx 2\frac{\gamma}{\hbar}\sqrt{n_0} \rightarrow \boxed{T = \frac{2\pi}{\omega} = \frac{\hbar\pi}{\gamma\sqrt{n_0}}}$$

Dephasing: $|\omega_{mean} t - \omega_{mean+1\,standard\,deviation} t| \simeq \frac{\pi}{2}$

$$\omega_{mean} = 2\frac{\gamma}{\hbar}\sqrt{n_0}$$

$$\omega_{mean+1\,standard\,deviation} = 2\frac{\gamma}{\hbar}\sqrt{n_0+\sqrt{n_0}} = 2\frac{\gamma}{\hbar}\sqrt{n_0}\sqrt{1+\frac{1}{\sqrt{n_0}}} \leftarrow n_0 \text{ large}$$

$$\approx 2\frac{\gamma}{\hbar}\sqrt{n_0}\left(1+\frac{1}{2\sqrt{n_0}}\right) = \omega_{mean} + \frac{\gamma}{\hbar}$$

$$|\omega_{mean} t - \omega_{mean+1\,standard\,deviation} t| = \frac{\gamma}{\hbar} t$$

$$\boxed{t = \frac{\pi\hbar}{2\gamma}}$$