Problem M17Q.2

(a) We first integrate

$$\int_{-\infty}^{\infty} \psi(q)^2 \,\mathrm{d}q = C^2 \sqrt{\frac{1}{\alpha}} \int_{-\infty}^{\infty} \exp\bigl(-u^2\bigr)\,\mathrm{d}u = C^2 \sqrt{\frac{\pi}{\alpha}},$$

where $u := q\sqrt{\alpha}$. Thus $C = (\alpha/\pi)^{1/4}$. Then

$$\begin{aligned}(\delta q)^2 &= \langle q^2 \rangle \\ &= \int_{-\infty}^{\infty} q^2\,\psi(q)^2\,\mathrm{d}q \\ &= \frac{1}{\alpha}\sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} u^2\,e^{-u^2}\,\mathrm{d}u \\ &= \frac{1}{2\alpha},\end{aligned}$$

and similarly

$$\begin{aligned}(\delta p)^2 &= \langle p^2 \rangle \\ &= -\hbar^2 \int_{-\infty}^{\infty} \psi(q)\partial_q^2\psi(q)\,\mathrm{d}q \\ &= \alpha\hbar^2 \sqrt{\frac{1}{\pi}} \int_{-\infty}^{\infty} (1-u^2)\,e^{-u^2}\,\mathrm{d}u \\ &= \frac{\alpha\hbar^2}{2}.\end{aligned}$$

We may verify that

$$(\delta p)\,(\delta q) = \frac{\hbar}{2}$$

as desired.

(b) In the Heisenberg picture, we have

$$-i\hbar\frac{\mathrm{d}}{\mathrm{d}t}q = [H,q] = \frac{1}{2m}\bigl[p^2,q\bigr] = \frac{-i\hbar}{m}p,$$

and similarly

$$-i\hbar\frac{\mathrm{d}}{\mathrm{d}t}p = [H,p] = \frac{1}{2}m\omega^2\bigl[q^2,p\bigr] = i\hbar m\omega^2 q.$$

Define the operator

$$z(t) := \alpha(t)\,q(t) + i\frac{p(t)}{\hbar}, \quad \text{so that} \quad z(0)\,\psi(q) = 0.$$

We directly compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}z(t) &= \alpha'(t)q(t) + \alpha(t)\frac{p(t)}{m} - \frac{im\omega^2}{\hbar}q(t) \\ &= -i\frac{\hbar}{m}\alpha(t)z(t) + \left[i\frac{\hbar}{m}\alpha(t)^2 + \alpha'(t) - \frac{im\omega^2}{\hbar}\right]q(t).\end{aligned}$$

Thus define the function

$$I(t) := \frac{i\hbar}{m}\int_0^t \alpha(s)\,\mathrm{d}s\,,$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{I(t)}z(t)\right] = e^{I(t)}\left[\frac{i\hbar}{m}\alpha(t)z(t) + z'(t)\right]$$
$$= e^{I(t)}\left[i\frac{\hbar}{m}\alpha(t)^2 + \alpha'(t) - \frac{im\omega^2}{\hbar}\right]q(t).$$

In particular, if

$$i\frac{\hbar}{m}\alpha(t)^2 + \alpha'(t) - \frac{im\omega^2}{\hbar} = 0, \qquad (*)$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{I(t)}z(t)\,\psi(q)\right] = 0,$$

forcing $z(t)\,\psi(q) = 0$ identically as desired. Letting $y := \alpha\hbar/m$, the differential equation (*) becomes

$$-y^2 + iy' + \omega^2 = 0, \qquad (**)$$

which always has a solution by the Picard-Lindelöf theorem.

(c) It remains to solve the differential equation (**). This equation is separable and rearranges to

$$-it = \int \frac{\mathrm{d}y}{y^2 - \omega^2} = -\tanh^{-1}(y/\omega)/\omega + c,$$

which rearranges to

$$y(t) = \omega\tanh(\omega\,(it + c)).$$

To match initial conditions, we choose

$$c := \omega^{-1}\tanh^{-1}\left(\alpha(0)\frac{\hbar}{m\omega}\right) > 0.$$

For $\mathrm{Re}\{z\} > 0$, recall that that $\tanh z = i\tan(-iz)$ is π-periodic in $\mathrm{Im}\{z\}$, and takes real values precisely when $2\,\mathrm{Im}\{z\} \in \pi\mathbb{Z}$.

Thus $\alpha(t)$ is real if and only if t is a multiple of $t_0 := \pi/(2\omega)$. For even multiples of t_0, we have $\alpha(t) = \alpha(0)$. For odd multiples of t_0, we have

$$y(t) = \omega\tanh\left(\omega c + i\frac{\pi}{2}\right) = \omega\coth(\omega c) = \frac{\omega}{\alpha(0)\hbar/m\omega},$$

so that

$$\alpha(t) = \boxed{\frac{m^2\omega^2}{\alpha(0)\hbar^2}.}$$

Thus the particle alternates between highly localized states for even multiples of t_0, and highly spread-out states at odd multiples of t_0.