1. **Dark Matter**

There is evidence that the visible stars in our galaxy move in a very diffuse background of dark matter. For simplicity, assume that in the neighborhood of a star, this dark matter has a mass distribution that is spherically symmetric about the star. Consider the effect of this extra matter on a planet orbiting around a star of mass M. The dark matter interacts with the planet only via gravitational attraction. Assume that near and inside the radius of planet's orbit, the mass density ρ of the dark matter is spatially uniform. Treat all motion and gravity nonrelativistically.

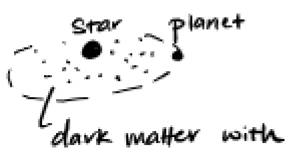
(a) Compute the angular frequency of a planet in circular orbit (radius R) about the star.

(b) Show that a nearly circular orbit will precess and compute the angular amount of precession per orbit. You may assume that the total mass of the dark matter inside a sphere of radius equal to that of the planet's orbit is small compared to the mass M of the star.

a) In general, gravitational force is given by $F = \frac{-GMm}{r^2}$

so for dark matter in the planet's orbit, we have $F = \frac{-G\overset{\text{dark matter}}{M_{DM}}m}{r^2}$

$$M_{DM} = \underset{\text{density}}{\rho}\ \underset{\text{volume of spherical region bounded by planet's orbit}}{\left(\tfrac{4}{3}\pi r^3\right)}$$



$$\Rightarrow F = -\frac{G\left(\frac{4}{3}\pi r^3\right)\rho m}{r^2} = -G\left(\tfrac{4}{3}\pi r\right)\rho m$$

To find potential energy V, use $F = \frac{-\partial V}{\partial r} \Rightarrow V = -\int F\,dr$

$$V = -\int\Big(\underbrace{\frac{-GMm}{r^2}}_{\text{gravitational force due to the star}} - G\left(\tfrac{4}{3}\pi r\right)\rho m\Big)dr = -\left[\frac{GMm}{r} - G\left(\tfrac{2}{3}\pi r^2\right)\rho m\right]$$

$$\Rightarrow V = \frac{-GMm}{r} + G\rho m\left(\tfrac{2}{3}\pi r^2\right)$$

Now writing the Lagrangian

$$\mathcal{L} = T - V = \underbrace{\tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)}_{T\text{ for orbital motion}} + \frac{GMm}{r} - G\rho m\left(\tfrac{2}{3}\pi r^2\right)$$

Euler-Lagrange equations

$$\phi: \quad \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 \quad \text{so} \quad \frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0$$

angular momentum $\ell = mr^2\dot{\phi}$ is conserved

r: $\quad \frac{\partial \mathcal{L}}{\partial r} = mr\dot{\phi}^2 - \frac{GMm}{r^2} - (\frac{4}{3}\pi r) G\rho m$

and $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\ddot{r}$

$$\Rightarrow m\ddot{r} = mr\dot{\phi}^2 - \frac{GMm}{r^2} - \frac{4}{3}\pi r G\rho m$$

for a circular orbit where $r = R$, $\ddot{r} = 0$

so $\quad 0 = mR\dot{\phi}^2 - \frac{GMm}{R^2} - \frac{4}{3}\pi R G\rho m$

here $\dot{\phi} = \omega_{orb}$, the angular frequency of the planet's orbit

Solving for ω_{orb}:

$$\cancel{m} R\omega_{orb}^2 = \frac{GM\cancel{m}}{R^2} + \frac{4}{3}\pi R G\rho \cancel{m}$$

$$\omega_{orb}^2 = \frac{GM}{R^3} + \frac{4}{3}\pi G\rho$$

$$\boxed{\omega_{orb} = \sqrt{\frac{GM}{R^3} + \frac{4}{3}\pi G\rho}}$$

Sanity Check:

Dark matter effectively adds mass to the star, so the planet experiences a greater gravitational force and orbits faster (given a fixed circular orbit)

without dark matter $\omega = \sqrt{\frac{GM}{R^3}} < \sqrt{\frac{GM}{R^3} + \frac{4}{3}\pi G\rho}$ ✓

b) nearly circular $\Rightarrow$ let $r = R + \delta$, where $\delta \ll R$ $\quad \Rightarrow \frac{\delta}{R} \ll 1$

because R is a constant $\dot{r} = \dot{\delta}$ and $\ddot{r} = \ddot{\delta}$

then returning to the E-L equations

$$m\ddot{\delta} = mr\dot{\phi}^2 - \frac{GMm}{r^2} - \frac{4}{3}\pi r G\rho m$$

replace $\dot{\phi}^2$ with l, which is constant, $\quad l = mr^2\dot{\phi} \Rightarrow \dot{\phi} = \frac{l}{mr^2}$

$$m\ddot{\delta} = mr\frac{l^2}{m^2r^4} - \frac{GMm}{r^2} - \frac{4}{3}\pi r G\rho m = \frac{l^2}{mr^3} - \frac{GMm}{r^2} - \frac{4}{3}\pi r G\rho m$$

$$m\ddot{\delta} = \frac{l^2}{m(R+\delta)^3} - \frac{GMm}{(R+\delta)^2} - \frac{4}{3}\pi (R+\delta) G\rho m$$

$$m\ddot{\delta} = \frac{l^2}{mR^3(1+\delta/R)^3} - \frac{GMm}{R^2(1+\delta/R)^2} - \frac{4}{3}\pi R G\rho m (1 + \frac{\delta}{R})$$

because $\frac{\delta}{R} \ll 1$, you can treat it as a small variable and Taylor expand

$$(1+x)^{-n} \approx 1 - nx + \cdots \quad \text{for } x \ll 1$$

$$\Rightarrow (1+\delta/R)^{-3} \approx 1 - 3(\tfrac{\delta}{R}) \quad ; \quad (1+\tfrac{\delta}{R})^{-2} \approx 1 - 2(\tfrac{\delta}{R})$$

$$m\ddot{\delta} = \frac{\ell^2}{mR^3}\left(1 - \frac{3\delta}{R}\right) - \frac{GMm}{R^2}\left(1 - \frac{2\delta}{R}\right) - \frac{4}{3}\pi R G\rho m\left(1 + \frac{\delta}{R}\right)$$

$$m\ddot{\delta} = \frac{\ell^2}{mR^3} - 3\delta\frac{\ell^2}{mR^4} - \frac{GMm}{R^2} + 2\delta\frac{GMm}{R^3} - \frac{4}{3}\pi R G\rho m - \delta\left(\frac{4}{3}\pi G\rho m\right)$$

$$m\ddot{\delta} = \underbrace{\left(\frac{\ell^2}{mR^3} - \frac{GMm}{R^2} - \frac{4}{3}\pi R G\rho m\right)}_{=0,\text{ this is just the circular orbit equation}} + \left(-\frac{3\ell^2}{mR^4} + \frac{2GMm}{R^3} - \frac{4}{3}\pi G\rho m\right)\delta$$

$$\ddot{\delta} = \left(-\frac{3\ell^2}{m^2R^4} + \frac{2GM}{R^3} - \frac{4}{3}\pi G\rho\right)\delta$$

this has the form $\ddot{\delta} = -\omega_{osc}^2\,\delta$, meaning the planet's orbit precesses with angular frequency ω_{osc}

$$-\omega_{osc}^2 = \frac{-3\ell^2}{m^2R^4} + \frac{2GM}{R^3} - \frac{4}{3}\pi G\rho$$

$$\omega_{osc}^2 = \frac{3\ell^2}{m^2R^4} - \frac{2GM}{R^3} + \frac{4}{3}\pi G\rho$$

to reduce further, notice $\ell = mr^2\dot{\phi} = mR^2\omega_{orb}$ from circular orbit (but since ℓ is constant we can replace it here despite the fact that the orbit is no longer circular)

$$\omega_{osc}^2 = \frac{3(m^2R^4\omega_{orb}^2)}{m^2R^4} - \frac{2GM}{R^3} + \frac{4}{3}\pi G\rho$$

$$\hookrightarrow\ 3\omega_{orb}^2 = 3\left(\frac{GM}{R^3} + \frac{4}{3}\pi G\rho\right) = \frac{3GM}{R^3} + 4\pi G\rho$$

$$\omega_{osc}^2 = \frac{3GM}{R^3} - \frac{2GM}{R^3} + 4\pi G\rho + \frac{4}{3}\pi G\rho$$

$$\omega_{osc} = \sqrt{\frac{GM}{R^3} + \frac{16}{3}\pi G\rho}$$

then the rate of precession is $\omega_{osc} - \omega_{orb} = \dot{\phi}$ ↦ not to be confused with the planet's orbital frequency!

and the angular amount of precession $\Delta\phi$ given by:

$$\frac{\Delta\phi}{T} = \dot{\phi} \qquad \Delta\phi = \dot{\phi}T \qquad T = \frac{2\pi}{\omega_{orb}} \rightarrow \text{period of planet's orbit}$$

$$\Delta\phi = \frac{2\pi(\omega_{osc} - \omega_{orb})}{\omega_{orb}}$$

$$\omega_{osc} = \sqrt{\frac{GM}{R^3} + \frac{16}{4}\pi G\rho} = \sqrt{\frac{GM}{R^3}}\sqrt{1 + \frac{16}{3}\frac{\pi\rho R^3}{3M}}$$

we are told the total mass of dark matter within the orbit is small compared to the star's mass

$$\Rightarrow \frac{4}{3}\pi R^3\rho \ll M, \text{ or } \frac{4\pi R^3\rho}{3M} \ll 1 \qquad \text{let } \alpha \equiv \frac{4\pi R^3\rho}{3M}, \text{ such that } \alpha \ll 1$$

and $\omega_{osc} = \sqrt{\frac{GM}{R^3}}\sqrt{1 + 4\alpha}$

Additionally, $\omega_{orb} = \sqrt{\frac{GM}{R^3}}\sqrt{1+\frac{4}{3}\frac{\pi R^3}{M}\rho} = \sqrt{\frac{GM}{R^3}}\sqrt{1+\alpha}$

Taylor expand around $\alpha \ll 1$

$$(1+4\alpha)^{1/2} \approx 1+\tfrac{1}{2}(4\alpha) = 1+2\alpha$$
$$(1+\alpha)^{1/2} \approx 1+\tfrac{1}{2}\alpha$$

$\Rightarrow$ $\omega_{osc} = \sqrt{\frac{GM}{R^3}}(1+2\alpha)$ and $\omega_{orb} = \sqrt{\frac{GM}{R^3}}(1+\frac{1}{2}\alpha)$

$$\Delta\phi \approx \frac{2\pi \cdot \sqrt{\frac{GM}{R^3}}\left(1+2\alpha-1-\frac{1}{2}\alpha\right)}{\sqrt{\frac{GM}{R^3}}\left(1+\frac{1}{2}\alpha\right)} = \frac{2\pi\left(\frac{3}{2}\alpha\right)}{1+\frac{1}{2}\alpha} = \frac{3\pi\alpha}{1+\frac{1}{2}\alpha}$$

$(1+\frac{1}{2}\alpha)^{-1} \approx 1-\frac{1}{2}\alpha$ $\Rightarrow$ $\Delta\phi \approx 3\pi\alpha(1-\frac{1}{2}\alpha) \approx 3\pi\alpha$ (neglecting α^2 terms)

So $\Delta\phi \approx 3\pi\left(\frac{4\pi R^3}{3M}\rho\right) = \boxed{\frac{4\pi^2 R^3\rho}{M}}$