1. Dark Matter

There is evidence that the visible stars in our galaxy move in a very diffuse background of dark matter. For simplicity, assume that in the neighborhood of a star, this dark matter has a mass distribution that is spherically symmetric about the star. Consider the effect of this extra matter on a planet orbiting around a star of mass $M$. The dark matter interacts with the planet only via gravitational attraction. Assume that near and inside the radius of planet’s orbit, the mass density $\rho$ of the dark matter is spatially uniform. Treat all motion and gravity nonrelativistically.

(a) Compute the angular frequency of a planet in circular orbit (radius $R$) about the star.

(b) Show that a nearly circular orbit will precess and compute the angular amount of precession per orbit. You may assume that the total mass of the dark matter inside a sphere of radius equal to that of the planet’s orbit is small compared to the mass $M$ of the star.

a) In general, gravitational force is given by $F = \frac{-GMm}{r^2}$

so for dark matter in the planet’s orbit, we have $F = \frac{-GM\rho m}{r^2}$

$\rho = \frac{\frac{4}{3}\pi r^3}{\text{volume of spherical region bounded by planet’s orbit}}$

$\Rightarrow F = \frac{-G\left(\frac{4}{3}\pi r^3\right)\rho m}{r^2} = -G\left(\frac{4}{3}\pi r^3\right)\rho m$

To find potential energy $V$, use $F = -\frac{\partial V}{\partial r} \Rightarrow V = -\int F \, dr$

$V = -\int \left(\frac{GMm}{r^2} - G\left(\frac{4}{3}\pi r^3\right)\rho m\right) dr = -\left[ \frac{GMm}{r} - G\left(\frac{4}{3}\pi r^3\right)\rho m \right]$

gravitational force due to the star

$\Rightarrow V = \frac{-GMm}{r} + G\rho m \left(\frac{2}{3}\pi r^2\right)$

Now writing the Lagrangian

$L = T - V = \frac{1}{2} m \left(l^2 + r^2 \dot{\phi}^2\right) + \frac{GMm}{r} - G\rho m \left(\frac{2}{3}\pi r^2\right)$

Enter Lagrange equations

$\dot{\phi}: \frac{\partial L}{\partial \phi} = 0 \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 0 \Rightarrow \frac{d}{dt} \left(\frac{1}{2} m r^2 \dot{\phi}^2\right) = 0$

so $\frac{d}{dt} (mr^2 \dot{\phi}) = 0$

angular momentum $l = mr^2 \dot{\phi}$ is conserved
\[
\frac{\partial x}{\partial r} = \frac{m_r \dot{r}^2}{r^2} - \frac{G M_m}{r^2} - \left(\frac{2}{3} \pi r\right) G \rho_m
\]

and \[
\frac{d}{dt} \frac{\partial x}{\partial r} = \dot{m}_r
\]

\[
\Rightarrow \dot{m}_r = m_r \dot{r}^2 - \frac{G M_m}{r^2} - \frac{4}{3} \pi r G \rho_m
\]

for a circular orbit where \(r = R\), \(\dot{r} = 0\)

so \(0 = m R \dot{r}^2 - \frac{G M_m}{R^2} - \frac{4}{3} \pi R G \rho_m\)

here \(\dot{r} = \omega R \), the angular frequency of the planet's orbit

Solving for \(\omega_{\text{orb}}\):

\[
\omega_{\text{orb}} = \sqrt{\frac{G M}{R^3} + \frac{4}{3} \pi R G \rho_m}
\]

**Sanity Check:**

Dark matter effectively adds mass to the star, so the planet experiences a greater gravitational force and orbits faster (given a fixed circular orbit) without dark matter \(\omega = \sqrt{\frac{G M}{R^3}} < \sqrt{\frac{G M}{R^3} + \frac{4}{3} \pi R G \rho_m}\) \(\checkmark\)

b) nearly circular \(\Rightarrow \) let \(r = R + \delta\), where \(\delta \ll R \Rightarrow \frac{\delta}{R} \ll 1\)

because \(R\) is a constant \(\dot{r} = \dot{s}\) and \(\dot{s} = \dot{\delta}\)

then returning to the E-L equations

\[
\dot{\delta} = m_r \dot{r}^2 - \frac{G M_m}{r^2} - \frac{4}{3} \pi r G \rho_m
\]

replace \(\dot{r}^2\) with \(l^2\), which is constant,

\(l = m_r \dot{r} \Rightarrow \dot{r} = \frac{l}{mr}\)

\[
\dot{\delta} = m_r \frac{l^2}{m^2 r^2} - \frac{G M_m}{r^2} - \frac{4}{3} \pi r G \rho_m
\]

\[
\dot{\delta} = \frac{l^2}{m (R+\delta)^3} - \frac{G M_m}{(R+\delta)^2} - \frac{4}{3} \pi (R+\delta) G \rho_m
\]

\[
\dot{\delta} = \frac{l^2}{m r^2 (1 + \frac{\delta}{R})^3} - \frac{G M_m}{r^2 (1 + \frac{\delta}{R})^2} - \frac{4}{3} \pi r G \rho_m (1 + \frac{\delta}{R})
\]

because \(\frac{\delta}{R} \ll 1\), you can treat it as a small variable and Taylor expand

\( \frac{1}{(1+\delta)} \approx 1 - \delta + \cdots \) for \(\delta \ll 1\)

\( (1+\frac{\delta}{R})^{-3} \approx 1 - 3 \left(\frac{\delta}{R}\right) + \cdots \approx 1 - 3 \left(\frac{\delta}{R}\right) \)

\( (1+\frac{\delta}{R})^{-2} \approx 1 - 2 \left(\frac{\delta}{R}\right) \)
\[ m \ddot{\delta} = \frac{\Delta \epsilon}{mR^3} \left( l - \frac{3\delta}{R} \right) - \frac{GMm}{R^2} \left( 1 - \frac{2\delta}{R} \right) - \frac{4}{3} \pi R \rho m \left( 1 + \frac{\delta}{R} \right) \]

\[ m \ddot{\delta} = \frac{\Delta \epsilon}{mR^3} - 3 \Delta \frac{\Delta \epsilon}{mR^3} \frac{GMm}{R^2} + 2 \Delta \frac{GMm}{R^3} - \frac{4}{3} \pi R \rho m \delta \left( \frac{4}{3} \pi \rho m \right) \]

\[ m \ddot{\delta} = \left( \frac{\Delta \epsilon}{mR^3} - \frac{GMm}{R^2} - \frac{4}{3} \pi R \rho m \right) \delta + \left( \frac{3\Delta \epsilon}{mR^3} + \frac{2GMm}{R^3} - \frac{4}{3} \pi R \rho m \right) \delta \]

\[ = 0, \text{ this is just the } \textit{circular orbit equation} \]

\[ \ddot{\delta} = \left( -\frac{3\Delta \epsilon}{mR^3} + \frac{2GMm}{R^3} - \frac{4}{3} \pi R \rho m \right) \delta \]

This has the form \( \ddot{\delta} = -\omega_{osc}^2 \delta \), meaning the planet's orbit precesses with angular frequency \( \omega_{osc} \)

\[ -\omega_{osc}^2 = -\frac{3\Delta \epsilon}{m^2R^3} + \frac{2GM}{R^3} - \frac{4}{3} \pi R \rho m \]

\[ \omega_{osc} = \frac{2\Delta \epsilon}{mR^3} - \frac{2GM}{R^3} + \frac{4}{3} \pi R \rho m \]

to reduce further, notice \( l = mR^2 \omega = mR^2 \omega_{orb} \) from circular orbit (but since \( l \) is constant we can replace it here despite the fact that the orbit is no longer circular)

\[ \omega_{osc} = \frac{3GM}{R^3} - \frac{2GM}{R^3} + \frac{4}{3} \pi R \rho m = \frac{2GM}{R^3} + \frac{4}{3} \pi R \rho m \]

\[ \omega_{osc} = \sqrt{\frac{2GM}{R^3} + \frac{16}{3} \pi R \rho m} \]

Then the rate of precession is \( \omega_{precess} = \dot{\delta} \) planet's orbital frequency

\[ \text{and the angular amount of precession } \Delta \phi \text{ given by:} \]

\[ \frac{\Delta \phi}{T} = \dot{\delta} \quad \Delta \phi = \dot{\delta} T \quad T = \frac{2\pi}{\omega_{orb}} \text{ period of planet's orbit} \]

\[ \Delta \phi = \frac{2\pi(\omega_{osc} - \omega_{precess})}{\omega_{orb}} \]

\[ \omega_{osc} = \sqrt{\frac{2GM}{R^3} + \frac{16}{3} \pi R \rho m} = \sqrt{\frac{GM}{R^3} \left[ 1 + \frac{16}{9} \pi \rho R^3 \right]} \]

we are told the total mass of dark matter within the orbit is small compared to the star's mass

\[ \Rightarrow \frac{4}{3} \pi R^3 \rho \ll M \text{ or } \frac{4}{3} \pi R^3 \rho \ll 1 \]

let \( \kappa \equiv \frac{4\pi R^3 \rho}{3M} \), such that \( \kappa \ll 1 \)

and \( \omega_{osc} = \sqrt{\frac{GM}{R^3} \left[ 1 + 4\kappa \right]} \)
Additionally, \( w_{orb} = \sqrt{\frac{GM}{R^3}} \sqrt{1 + \frac{4}{3} \frac{mR^3}{M}} = \sqrt{\frac{GM}{R^3}} \sqrt{1 + \alpha} \)

Taylor expand around \( \alpha = 1 \)

\[
(1 + 4\alpha)^{1/2} \approx 1 + \frac{1}{2}(4\alpha) = 1 + 2\alpha \\
(1 + \alpha)^{1/2} \approx 1 + \frac{1}{2}\alpha
\]

\[\Rightarrow w_{osc} = \sqrt{\frac{GM}{R^3}} (1 + 2\alpha) \quad \text{and} \quad w_{orb} = \sqrt{\frac{GM}{R^3}} (1 + \frac{1}{2}\alpha)\]

\[
\Delta \phi \approx \frac{2\pi \cdot \sqrt{\frac{GM}{R^3}} (1 + 2\alpha - 1 - \frac{1}{2}\alpha)}{\sqrt{\frac{GM}{R^3}} (1 + \frac{1}{2}\alpha)} = \frac{2\pi (\frac{3}{2}\alpha)}{1 + \frac{1}{2}\alpha} = \frac{3\pi\alpha}{1 + \frac{1}{2}\alpha}
\]

\[(1 + \frac{1}{2}\alpha)^{-1} \approx 1 - \frac{1}{2}\alpha \quad \Rightarrow \quad \Delta \phi \approx 3\pi\alpha(1 - \frac{1}{2}\alpha) \approx 3\pi\alpha \quad (\text{neglecting} \ \alpha^2 \ \text{terms})
\]

So \[\Delta \phi \approx 3\pi \left( \frac{4\pi^2 R^3 \rho}{3M} \right) = \frac{4\pi^2 R^3 \rho}{M}\]