Problem M17M.2

(a) Let $\theta$ denote the angle of the stick from the vertical. Then the kinetic energy of the stick is

$$T := \frac{1}{2} \int_0^L \lambda(x) (x \dot{\theta})^2 \, dx,$$

and its potential energy is

$$U := g \int_0^L \lambda(x) (x \cos \theta) \, dx.$$

We may write these as

$$T = \frac{M_2}{2} \dot{\theta}^2 \quad \text{and} \quad U = M_1 g \cos \theta,$$

where $M_\alpha := \int_0^L \lambda(x) x^\alpha \, dx$.

The overall vertical momentum of the stick is

$$P = \int_0^L \lambda(x) (x \dot{\theta} \sin \theta) \, dx = M_1 \dot{\theta} \sin \theta,$$

where positive values are in the downward direction. Then Newton’s second law states that

$$\dot{P} = M_0 g - F_{\text{normal}},$$

where $F_{\text{normal}}$ is the normal force applied by the table. Thus the stick leaves the table at the moment when $\dot{P} > M_0 g$.

Conservation of energy gives the equation of motion

$$\frac{M_2}{2} \ddot{\theta}^2 + M_1 g \cos \theta = M_1 g \cos \theta_0,$$

and differentiating yields

$$M_2 \ddot{\theta} = M_1 g \sin \theta.$$

It follows that

$$\dot{P} = M_1 \dot{\theta} \sin \theta + M_1 \dot{\theta}^2 \cos \theta$$

$$= \frac{M_1^2}{M_2} g \sin^2 \theta + \frac{2 M_1^2}{M_2} g (\cos \theta_0 - \cos \theta) \cos \theta,$$

so the stick leaves the table precisely when

$$\sin^2 \theta + 2 (\cos \theta_0 - \cos \theta) \cos \theta > \frac{M_0 M_2}{M_1^2}.$$

Letting $t := \cos \theta$, this condition can be written as

$$3t^2 - 2t \cos \theta_0 + \left( \frac{M_0 M_2}{M_1^2} - 1 \right) < 0.$$

If the stick never leaves the table, then the left-hand side must be non-negative for all $0 \leq t \leq 1$. Differentiating the left-hand side, the expression is minimized when

$$t = \frac{\cos \theta_0}{3},$$

which is always within the range $0 \leq t \leq 1$. Thus the minimum value of the quadratic is

$$-\frac{\cos^2 \theta_0}{3} + \left( \frac{M_0 M_2}{M_1^2} - 1 \right).$$
and the stick maintains contact with the table if and only if

\[ \frac{M_0M_2}{M_1^2} \geq 1 + \frac{\cos^2 \theta_0}{3}. \]

Equivalently, note that the location of the center of mass of the stick is \( x_{CM} = M_1/M_0 \), so its moment of inertia is

\[
I_{CM} = \int_0^L \lambda(x) (x - x_{CM})^2 \, dx = M_2 - 2x_{CM}M_1 + M_0x_{CM}^2
\]

\[
= M_2 - 2 \frac{M_1^2}{M_0} + \frac{M_1^2}{M_0}
\]

\[
= M_2 - \frac{M_1^2}{M_0}.
\]

Thus the condition can also be written

\[ \cos^2 \theta_0 \leq \frac{3I_{CM}}{M_0x_{CM}^2}. \]

(b) For the given mass distribution, we compute

\[
M_0 = \int_0^L \lambda(x) \, dx = \frac{L^3}{6}, \quad M_1 = \frac{L^4}{12}, \quad \text{and} \quad M_2 = \frac{L^5}{20},
\]

so that

\[ \frac{M_0M_2}{M_1^2} = \frac{6}{5}. \]

From part (1), the stick maintains contact with the table if and only if

\[ \frac{\cos^2 \theta_0}{3} + 1 \leq \frac{6}{5}, \]

which rearranges to

\[ \theta_0 \geq \cos^{-1} \left( \frac{3}{\sqrt{5}} \right). \]

Time: 39 m 32 s