Problem M17M.2

(a) Let θ denote the angle of the stick from the vertical. Then the kinetic energy of the stick is

$$T := \frac{1}{2}\int_0^L \lambda(x)\,(x\dot{\theta})^2\,\mathrm{d}x\,,$$

and its potential energy is

$$U := g\int_0^L \lambda(x)\,(x\cos\theta)\,\mathrm{d}x\,.$$

We may write these as

$$T = \frac{M_2}{2}\,\dot{\theta}^2 \quad \text{and} \quad U = M_1 g\cos\theta, \quad \text{where} \quad M_\alpha := \int_0^L \lambda(x)\,x^\alpha\,\mathrm{d}x\,.$$

The overall vertical momentum of the stick is

$$P = \int_0^L \lambda(x)(x\dot{\theta}\sin\theta)\,\mathrm{d}x = M_1\dot{\theta}\sin\theta,$$

where positive values are in the downward direction. Then Newton's second law states that

$$\dot{P} = M_0\,g - F_{\text{normal}},$$

where F_{normal} is the normal force applied by the table. Thus the stick leaves the table at the moment when $\dot{P} > M_0 g$.

Conservation of energy gives the equation of motion

$$\frac{M_2}{2}\dot{\theta}^2 + M_1 g\cos\theta = M_1 g\cos\theta_0,$$

and differentiating yields

$$M_2\ddot{\theta} = M_1 g\sin\theta.$$

It follows that

$$\begin{aligned}\dot{P} &= M_1\ddot{\theta}\sin\theta + M_1\dot{\theta}^2\cos\theta \\ &= \frac{M_1^2}{M_2}g\sin^2\theta + \frac{2M_1^2}{M_2}g(\cos\theta_0 - \cos\theta)\,\cos\theta,\end{aligned}$$

so the stick leaves the table precisely when

$$\sin^2\theta + 2\,(\cos\theta_0 - \cos\theta)\cos\theta > \frac{M_0 M_2}{M_1^2}.$$

Letting $t := \cos\theta$, this condition can be written as

$$3t^2 - 2t\cos\theta_0 + \left(\frac{M_0 M_2}{M_1^2} - 1\right) < 0.$$

If the stick *never* leaves the table, then the left-hand side must be non-negative for all $0 \le t \le 1$. Differentiating the left-hand side, the expression is minimized when

$$t = \frac{\cos\theta_0}{3},$$

which is always within the range $0 \le t \le 1$. Thus the minimum value of the quadratic is

$$-\frac{\cos^2\theta_0}{3} + \left(\frac{M_0 M_2}{M_1^2} - 1\right),$$

and the stick maintains contact with the table if and only if

$$\boxed{\frac{M_0 M_2}{M_1^2} \geq 1 + \frac{\cos^2\theta_0}{3}.}$$

Equivalently, note that the location of the center of mass of the stick is $x_{CM} = M_1/M_0$, so its moment of inertia is

$$\begin{aligned} I_{CM} = \int_0^L \lambda(x)\,(x - x_{CM})^2\,\mathrm{d}x &= M_2 - 2x_{CM}M_1 + M_0 x_{CM}^2 \\ &= M_2 - 2\frac{M_1^2}{M_0} + \frac{M_1^2}{M_0} \\ &= M_2 - \frac{M_1^2}{M_0}. \end{aligned}$$

Thus the condition can also be written

$$\boxed{\cos^2\theta_0 \leq \frac{3I_{CM}}{M_0 x_{CM}^2}.}$$

(b) For the given mass distribution, we compute

$$M_0 = \int_0^L \lambda(x)\,\mathrm{d}x = \frac{L^3}{6}, \qquad M_1 = \frac{L^4}{12}, \quad \text{and} \quad M_2 = \frac{L^5}{20},$$

so that

$$\frac{M_0 M_2}{M_1^2} = \frac{6}{5}.$$

From part (1), the stick maintains contact with the table if and only if

$$\frac{\cos^2\theta_0}{3} + 1 \leq \frac{6}{5},$$

which rearranges to

$$\boxed{\theta_0 \geq \cos^{-1}\left(\sqrt{\frac{3}{5}}\right).}$$

Time: 39 m 32 s