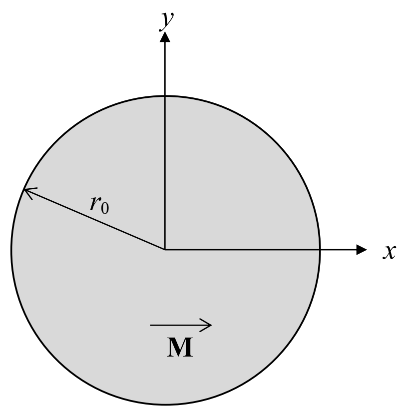
2 Jan 2022



2. **Moving Bar Magnet**




**(a)** A long cylinder of radius $r_0$ has uniform permanent magnetization density $\mathbf{M}$ perpendicular to the axis of the cylinder. Find the fields $\mathbf{B}$ and $\mathbf{H}$ everywhere. Let $\hat{z}$ =axis of the cylinder, and $\mathbf{M} = M\hat{x}$ as shown.

**(b)** Suppose the cylinder is given uniform velocity $\mathbf{v} = v\hat{z}$ along its axis. Find the resulting charge density and electric field everywhere. You may ignore effects of order $(v/c)^2$.

a) No free currents $\rightarrow$ magnetic scalar potential $\vec{B} = -\nabla u$

$$\vec{H}_{in} = \frac{1}{\mu_0}\vec{B}_{in} - \vec{M} \leftarrow \vec{M} = M\hat{x} = M\cos\theta\,\hat{s} - M\sin\theta\,\hat{\theta}, \quad \vec{B} = -\nabla u$$

$$= -\left(\frac{1}{\mu_0}\frac{\partial u_{in}}{\partial s}\hat{s} + \frac{1}{\mu_0 s}\frac{\partial u_{in}}{\partial \theta}\hat{\theta} + M\cos\theta\,\hat{s} - M\sin\theta\,\hat{\theta}\right)$$

$$\vec{H}_{out} = \frac{1}{\mu_0}\vec{B}_{out} = -\left(\frac{1}{\mu_0}\frac{\partial u_{out}}{\partial s}\hat{s} + \frac{1}{\mu_0 s}\frac{\partial u_{out}}{\partial \theta}\hat{\theta}\right)$$

$$\vec{K}_B = \hat{M} \times \hat{n} = (M\cos\theta\,\hat{s} - M\sin\theta\,\hat{\theta}) \times \hat{s} = M\sin\theta\,\hat{z}$$

Boundary Conditions at $s = s_0$

$$H^{\perp}_{out} - H^{\perp}_{in} = -(M^{\perp}_{out} - M^{\perp}_{in}) = M^{\perp}_{in} = M\cos\theta$$

$$\vec{H}^{\parallel}_{out} - \vec{H}^{\parallel}_{in} = \hat{K}_f \times \hat{n} = (\vec{0}) \times \hat{s} = \vec{0}$$

General Solution:

$$u_{out}(s,\theta) = A_0 + B_0 \ln s + \sum_{l=1}^{\infty} s^l (A_l \cos(l\theta) + B_l \sin(l\theta)) + r^{-l}(C_l \cos(l\theta) + D_l \sin(l\theta))$$

$\downarrow \leftarrow$ Bounded at $\infty$, $A_l, B_l, B_0 = 0$; constants don't change field, $A_0 = 0$; $l = 1$ from $\vec{M}$, $C_{l\neq 1}, D_{l\neq 1} = 0$

$$= C_1 \frac{1}{s}\cos(\theta) + D_1 \frac{1}{s}\sin(\theta)$$

$$u_{in}(s,\theta) = A_0 + B_0 \ln s + \sum_{l=1}^{\infty} s^l (A_l \cos(l\theta) + B_l \sin(l\theta)) + r^{-l}(C_l \cos(l\theta) + D_l \sin(l\theta))$$

$\downarrow \leftarrow$ Bounded at 0, $C_l, D_l, B_0 = 0$; constants don't change field, $A_0 = 0$; $l = 1$ from $\vec{M}$, $A_{l\neq 1}, B_{l\neq 1} = 0$

$$= A_1 s\cos(\theta) + B_1 s\sin(\theta)$$

$$H^{\perp}_{out} - H^{\perp}_{in} = H_{out,s} - H_{in,s} = -\frac{1}{\mu_0}\frac{\partial u_{out}}{\partial s} + \frac{1}{\mu_0}\frac{\partial u_{in}}{\partial s} + M\cos\theta = M\cos\theta$$

$$\left.\frac{\partial u_{out}}{\partial s}\right|_{r_0} = \left.\frac{\partial u_{in}}{\partial s}\right|_{r_0} \rightarrow A_1\cos\theta + B_1\sin\theta = -\frac{1}{r_0^2}(C_1\cos\theta + D_1\sin\theta)$$

Valid for all angles: $C_1 = -r_0^2 A_1$, $D_1 = -r_0^2 B_1$

$$\vec{H}^{\parallel}_{out} - \vec{H}^{\parallel}_{in} \rightarrow H_{out,\theta} - H_{in,\theta} = -\frac{1}{\mu_0 r_0}\frac{\partial u_{out}}{\partial \theta} + \frac{1}{\mu_0 r_0}\frac{\partial u_{in}}{\partial \theta} - M\sin\theta = 0$$

$$\frac{\partial u_{out}}{\partial \theta}\Big|_{r_0} - \frac{\partial u_{in}}{\partial \theta}\Big|_{r_0} = -M\mu_0 r_0 \sin\theta$$

$$\frac{\partial u_{out}}{\partial \theta}\Big|_{r_0} = -r_0\left(-A_1 \sin\theta + B_1 \cos\theta\right)$$

$$\frac{\partial u_{in}}{\partial \theta}\Big|_{r_0} = r_0\left(-A_1 \sin\theta + B_1 \cos\theta\right)$$

$$-M\mu_0 r_0 \sin\theta = -2r_0\left(-A_1 \sin\theta + B_1 \cos\theta\right)$$

Must be valid for all $\theta \rightarrow B_1 = 0,\ A_1 = -\frac{1}{2}M\mu_0$

$$U_{out}(s,\theta) = \frac{1}{2}\frac{M\mu_0 r_0^2}{s}\cos(\theta)$$

$$U_{in}(s,\theta) = -\frac{1}{2}M\mu_0 s\cos(\theta)$$

$$\vec{B}_{in} = -\left(\frac{\partial u_{in}}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial u_{in}}{\partial \theta}\hat{\theta}\right) = -\mu_0\left(-\frac{1}{2}M\cos\theta\,\hat{s} + \frac{1}{2}M\sin\theta\,\hat{\theta}\right) = \frac{1}{2}M\mu_0\left(\cos\theta\,\hat{s} - \sin\theta\,\hat{\theta}\right) = \frac{1}{2}M\mu_0\hat{x}$$

$$\vec{H}_{in} = \frac{1}{\mu_0}\vec{B}_{in} - \vec{M} = \frac{1}{2}M\hat{x} - M\hat{x} = -\frac{1}{2}M\hat{x}$$

$$\vec{B}_{out} = -\left(\frac{\partial u_{out}}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial u_{out}}{\partial \theta}\hat{\theta}\right) = -\frac{M\mu_0 r_0^2}{2}\left(-\frac{1}{s^2}(\cos\theta)\hat{s} + \frac{1}{s^2}(-\sin\theta)\hat{\theta}\right) = \frac{M\mu_0 r_0^2}{2s^2}\left(\cos\theta\,\hat{s} + \sin\theta\,\hat{\theta}\right)$$

$$\vec{H}_{out} = \frac{1}{\mu_0}\vec{B}_{out} = \frac{1}{2}\frac{M r_0^2}{s^2}\left(\cos\theta\,\hat{s} + \sin\theta\,\hat{\theta}\right)$$

$$\cos\theta\,\hat{s} + \sin\theta\,\hat{\theta} = \hat{x} + 2\sin\theta\,\hat{\theta} \leftarrow \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$= (1 - 2\sin^2\theta)\hat{x} + 2\sin\theta\cos\theta\,\hat{y} = (1 - 2\sin^2\theta)\hat{x} + \sin(2\theta)\hat{y}$$

$$\boxed{\vec{B} = \begin{cases} \frac{1}{2}M\mu_0\hat{x} & ,\ in \\ \frac{M\mu_0 r_0^2}{2s^2}\left(\cos\theta\,\hat{s} + \sin\theta\,\hat{\theta}\right) & ,\ out \end{cases} \quad \vec{H} = \begin{cases} -\frac{1}{2}M\hat{x} & ,\ in \\ \frac{1}{2}\frac{M r_0^2}{s^2}\left(\cos\theta\,\hat{s} + \sin\theta\,\hat{\theta}\right) & ,\ out \end{cases}}$$

b) $\beta = \frac{v}{c},\ \beta^2 \approx 0,\ \gamma = (1-\beta^2)^{-1/2} \approx 1$

$$\Lambda^{\mu'}_{\mu} = \begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & 1 \end{pmatrix},\ F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix},\ K^{\mu} = \begin{pmatrix} \sigma c \\ K_x \\ K_y \\ K_z \end{pmatrix}$$

$$\sigma' = \Lambda^{\sigma'}_{\mu} K^{\mu}/c = (\Lambda^{\sigma'}_{0} K^0 + \Lambda^{\sigma'}_{1} K^1 + \Lambda^{\sigma'}_{2} K^2 + \Lambda^{\sigma'}_{3} K^3)/c = (\sigma c - \beta K_z)/c = -K_{B,z}/c$$
$$= -\frac{M\beta}{c}\sin\theta = -M\frac{v}{c^2}\sin\theta$$

$$\boxed{\sigma' = -M\frac{v}{c^2}\sin\theta}$$

Both in and out, $\vec{E} = 0$ and $B_z = 0 \to$

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_y \\ 0 & 0 & 0 & B_x \\ 0 & B_y & -B_x & 0 \end{pmatrix}$$

$$F^{\mu'\nu'} = \Lambda^{\mu'}_{\mu}\Lambda^{\nu'}_{\nu}F^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_y \\ 0 & 0 & 0 & B_x \\ 0 & B_y & -B_x & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & -\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ \beta B_y & 0 & 0 & -B_y \\ -\beta B_x & 0 & 0 & B_x \\ 0 & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\beta B_y & \beta B_x & 0 \\ \beta B_y & 0 & 0 & -B_y \\ -\beta B_x & 0 & 0 & B_x \\ 0 & B_y & -B_x & 0 \end{pmatrix} \to \vec{E}'/c = \begin{pmatrix} \beta B_y \\ -\beta B_x \\ 0 \end{pmatrix}$$

$$\vec{E}'_{in} = \frac{v}{c^2}\left[(0)\hat{x} - \left(\frac{1}{2}M\mu_0\right)\hat{y}\right] = -\frac{1}{2}\frac{v}{c^2}M\mu_0\hat{y}$$
$$\vec{E}'_{out} = \frac{v}{c^2}\frac{M\mu_0 r_0^2}{2s^2}\left[(\sin(2\theta))\hat{x} - (1-2\sin^2\theta)\hat{y}\right]$$
$$= \frac{v}{c^2}\frac{M\mu_0 r_0^2}{2s^2}\left(\sin 2\theta\,\hat{x} - (1-2\sin^2\theta)\hat{y}\right)$$

$$\boxed{\vec{E}' = \begin{cases} -\frac{1}{2}\frac{v}{c^2}M\mu_0\hat{y} & \text{, in} \\ \frac{v}{c^2}\frac{M\mu_0 r_0^2}{2s^2}\left(\sin 2\theta\,\hat{x} - (1-2\sin^2\theta)\hat{y}\right) & \text{, out} \end{cases}}$$