12 Dec 2021

Section B. Electricity and Magnetism

1. **Plasma Waves**

   A 'tenuous plasma" consists of free electrons with mass m and charge e. There are n electrons per unit volume. Assume that the electron density is uniform and that interactions between the electrons may be neglected. Electromagnetic waves (frequency ω and wave number k) are incident on the plasma.

   (a) Find the conductivity σ of the plasma as a function of ω.

   (b) Find the dispersion relation; i.e. find the relation between k and ω.

   (c) Find the index of refraction as a function of ω. What does it tell you about the speed of wave propagation in the plasma? The plasma frequency is defined as $\omega_p^2 = ne^2/m\epsilon_0$ (in SI units). What happens if $\omega < \omega_p$?

M17E.1

a) Find σ : $\vec{J} = \sigma \vec{E}$

$\vec{J} = \rho \vec{v} = n e \vec{v}$

$\vec{E} = \frac{1}{e}\vec{F} = \frac{m}{e}\dot{\vec{v}} \Leftarrow \vec{E} = \vec{E}_0 e^{i(kx-\omega t)}$ s.t. "$\int \vec{E} dt$" $= \frac{-1}{i\omega}\vec{E}$

$= (-i\omega)\frac{m}{e}\vec{v} = \frac{m\omega}{ie}\vec{v}$

$n e \vec{v} = \sigma \frac{m\omega}{ie}\vec{v}$

$$\boxed{\sigma = \frac{ne^2}{m\omega} i}$$

b) Start from Maxwell's Eqns.

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \Leftarrow \vec{J} = \sigma \vec{E},\ \mu_0 \varepsilon_0 = \frac{1}{c^2}$

$= \mu_0 \sigma \vec{E} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \Leftarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\mu_0 \sigma \frac{\partial \vec{B}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} \Leftarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B})^{\to 0} - \nabla^2 \vec{B} = -\nabla^2 \vec{B}$

$\nabla^2 \vec{B} = \mu_0 \sigma \frac{\partial \vec{B}}{\partial t} + \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} \Leftarrow \vec{B} = \vec{B}_0 e^{i(kx-\omega t)}$

$-k^2 \vec{B} = -\mu_0 \sigma i\omega \vec{B} - \frac{1}{c^2}\omega^2 \vec{B}$

$k^2 = i\mu_0 \sigma \omega + \frac{\omega^2}{c^2} \Leftarrow \sigma = \frac{ne^2}{m\omega} i$

$= -\frac{ne^2 \mu_0 c^2}{m} \cdot \frac{1}{c^2} + \frac{\omega^2}{c^2} \Leftarrow \frac{ne^2 \mu_0 c^2}{m} = \frac{ne^2}{\varepsilon_0 m} = \omega_p^2$

$= \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}$

$$\boxed{(kc)^2 = \omega^2 - \omega_p^2}$$

c) index of refraction $\equiv I$ (n is already used in problem)

$$v_{phase} = \frac{c}{I} \leftarrow v_{phase} = \frac{\omega}{k}$$

$$I = \frac{ck}{\omega} \leftarrow k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}$$

$$= \frac{1}{\omega}\sqrt{\omega^2 - \omega_p^2} = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}$$

$$\boxed{I(\omega) = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}}$$

$I(\omega) < 1$ for all transmitted waves

$$v_{prop} = v_{phase} = \frac{c}{I} > c$$

$$\boxed{v_{prop} > c}$$

$\omega < \omega_p \rightarrow I(\omega)$ becomes imaginary

imaginary $I \rightarrow$ $\boxed{\text{absorption}}$