Section A. Mechanics

1. **Dark Matter**

   There is evidence that the visible stars in our galaxy move in a very diffuse background of dark matter. For simplicity, assume that in the neighborhood of a star, this dark matter has a mass distribution that is spherically symmetric about the star. Consider the effect of this extra matter on a planet orbiting around a star of mass M. The dark matter interacts with the planet only via gravitational attraction. Assume that near and inside the radius of planet's orbit, the mass density ρ of the dark matter is spatially uniform. Treat all motion and gravity nonrelativistically.

   (a) Compute the angular frequency of a planet in circular orbit (radius R) about the star.

   (b) Show that a nearly circular orbit will precess and compute the angular amount of precession per orbit. You may assume that the total mass of the dark matter inside a sphere of radius equal to that of the planet's orbit is small compared to the mass M of the star.

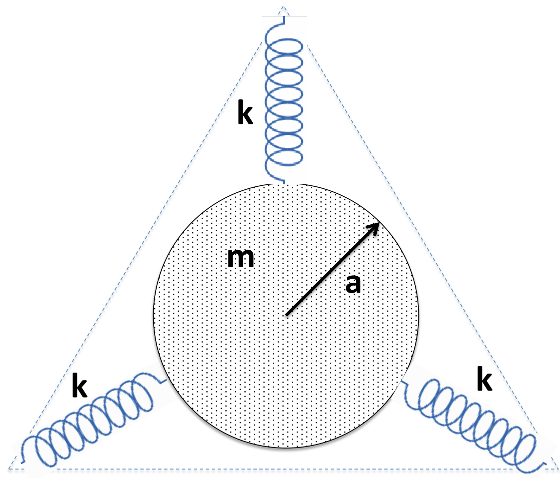
2. **Falling Rod**

A thin stick of length L with some mass distribution $\lambda(x)$ along it is initially at rest. It has one end ($x = 0$) on a horizontal table and initially makes an angle θ_0 with the vertical. Assume that the stick-table contact point has an infinite coefficient of friction (so that the end of the stick can lift off the table, but cannot slide on it). The stick is released from rest and allowed to fall to the table.

(a) Find the condition that the end of the stick initially in contact with the table does *not* rise from the table as the stick falls. Express the condition in terms of θ_0 and the mass distribution along the stick.

(b) For a stick of mass concentrated in the middle $\lambda(x) \propto (L/2)^2 - (x - L/2)^2$, what range of angles θ_0 keeps the stick in contact with the table throughout its fall?

3. **Disk with Three Springs**



A uniform disk of mass m and radius a rests on a horizontal frictionless surface. It is symmetrically attached to three identical, ideal, massless springs of spring constant k whose other ends are attached to the three vertices of an equilateral triangle.

At equilibrium, the length ℓ of the springs is greater than their relaxed length ℓ_0. The disk remains in the initial horizontal plane but is otherwise free to move. (The diagram shows the view looking down on the plane.) What are the frequencies of the normal modes of small oscillations? What do the modes look like?

Section B. Electricity and Magnetism

1. **Plasma Waves**

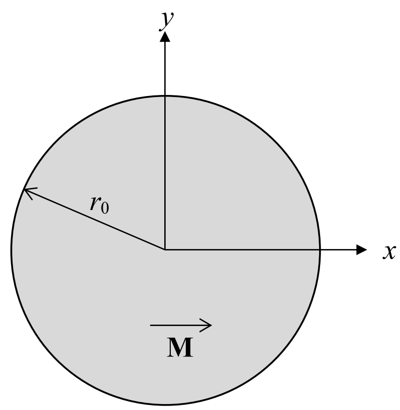
   A 'tenuous plasma" consists of free electrons with mass m and charge e. There are n electrons per unit volume. Assume that the electron density is uniform and that interactions between the electrons may be neglected. Electromagnetic waves (frequency ω and wave number k) are incident on the plasma.

   (a) Find the conductivity σ of the plasma as a function of ω.

   (b) Find the dispersion relation; i.e. find the relation between k and ω.

   (c) Find the index of refraction as a function of ω. What does it tell you about the speed of wave propagation in the plasma? The plasma frequency is defined as $\omega_p^2 = ne^2/m\epsilon_0$ (in SI units). What happens if $\omega < \omega_p$?

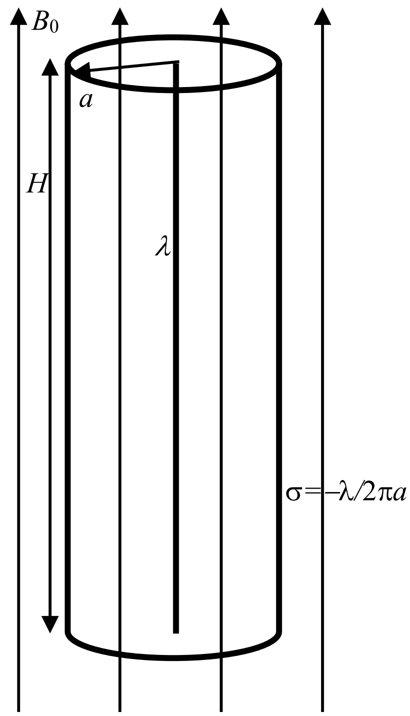
2. **Moving Bar Magnet**



(a) A long cylinder of radius r_0 has uniform permanent magnetization density $\mathbf{M}$ perpendicular to the axis of the cylinder. Find the fields $\mathbf{B}$ and $\mathbf{H}$ everywhere. Let $\hat{z}$ =axis of the cylinder, and $\mathbf{M} = M\hat{x}$ as shown.

(b) Suppose the cylinder is given uniform velocity $\mathbf{v} = v\hat{z}$ along its axis. Find the resulting charge density and electric field everywhere. You may ignore effects of order $(v/c)^2$.

3. **Spinning Cylinder**



A cylindrical capacitor consists of a line of charge with linear charge density λ and a concentric insulating tube of radius a with a compensating uniform surface charge density $\sigma = -\lambda/2\pi a$ on its surface (fixed, not free to move). The height of the capacitor $H \gg a$ so that you can ignore edge effects. The capacitor is placed in a uniform external magnetic field of strength B_0 parallel to the cylinder axis and pointing up. The insulating tube is free to rotate around its axis and its mass is all concentrated on the rim.

(a) Find the magnitude and direction of the electromagnetic angular momentum stored in the EM field.

(b) The external magnetic field B_0 is very slowly ramped down. Show that this will cause the tube to rotate and find the angular velocity of rotation when the *e*xternal B field is completely turned off.

Section A. Quantum Mechanics

1. **Atom in a Cavity**

Consider a simple model of a cavity mode of the electromagnetic field interacting with a two-state system representing an atom which can absorb or emit cavity mode photons. The Hamiltonian for this system is taken to be the sum of a harmonic oscillator representing the cavity mode, a term which splits the energy of the two-state system, and an interaction term:

$$H_0 = \epsilon_c a^\dagger a + \frac{1}{2}\epsilon_a \sigma_z + \gamma(a\sigma_+ + a^\dagger \sigma_-) \ ,$$

where $a^\dagger$ is the usual harmonic oscillator creation operator, ϵ_c is the energy of a cavity mode photon, ϵ_a is the energy difference between the ground (g) and excited (e) states of the two-state system, and the σ matrices act on the two-state system spanned by $\{|g\rangle, |e\rangle\}$.

(a) Show that this Hamiltonian can be block diagonalized in two-state subspaces spanned by the states $|g, n+1\rangle, |e, n\rangle$ (where n is the level of the harmonic oscillator, i.e. the number of cavity photons). Diagonalize this Hamiltonian to obtain an expression for the splitting between the two energy eigenstates in each block. Show that this splitting is smallest when the atom and the radiation are in resonance: $\epsilon_a = \epsilon_c$.

For the rest of this problem let $\epsilon_a = \epsilon_c$, so we are at resonance.

(b) Suppose that at time $t = 0$ the system is in the state $|e, n\rangle$. Because this is not an energy eigenstate, the state will evolve in time. Derive expressions for $p_g(t)$ and $p_e(t)$ (the probabilities of finding the atom in its ground and excited states respectively), as a function of time. Show that these probabilities undergo oscillations with a definite period.

(c) Suppose that at $t = 0$ the atom is in the excited state and the cavity is in a superposition of cavity modes: $\sum p_n |n>$ where $p(n) = (1/\sqrt{2\pi n_0}) \exp((n - n_0)^2/2n_0)$. When n_0 is large, this is pretty close to a *coherent state* of excitation of the cavity, which is as close as we get in quantum mechanics to a classical state of the field. Derive formal expressions for $p_e(t)$ and $p_g(t)$. For n_0 large, with what period do these probabilities initially oscillate? On roughly what time scale do these initial oscillations first dephase?

2. **Squeezing the Harmonic Oscillator**

The quantum harmonic oscillator, described by the Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$, with $[p, q] = \frac{\hbar}{i}$, is used to describe a remarkably diverse set of physical systems. Consider the class of harmonic oscillator 'squeezed state' wave functions of the form

$$\psi(q) = C \exp(-\alpha q^2/2)$$

where C is a normalization constant and α is an arbitrary *complex* number (with positive real part to ensure that the wave function is normalizable). States of this type can be prepared from the ground state by turning on a suitable interaction hamiltonian for the right length of time. It is not in general an energy eigenstate, but it has interesting and useful properties:

(a) The means of p and q in this state vanish. Calculate their variances and show that for *real* α this is a minimum uncertainty state, *i.e.* a state with $(\delta p)(\delta q) = \hbar/2$ (define $(\delta q)^2 = \langle (q - \langle q \rangle)^2 \rangle$, and similarly for δp).

(b) This state satisfies the equation

$$(\alpha q + ip/\hbar)\psi(q) = (\alpha q + \frac{\partial}{\partial q})\psi(q) = 0$$

Show that the time-evolved state $\psi_t(q) = \exp(-iH_0 t/\hbar)\psi(q)$ satisfies the same equation with a time-dependent (and in general complex) $\alpha(t)$. One way to do this is to go to the Heisenberg picture and time-evolve the operators p and q, rather than $\psi(q)$.

(c) You should find that $\alpha(t)$ becomes complex with time, even if $\alpha(0) = \alpha$ is real. So, in general, at $t > 0$ the state is no longer minimum uncertainty. Show that there are later times in the harmonic oscillator period where $\alpha(t)$ becomes real again (and the state recovers its minimum uncertainty character). Identify these real values of $\alpha(t)$ and comment on the nature of the minimum uncertainty states that are visited (assume that α is large, so that the initial minimum uncertainty state has small variance in q (and large variance in p).

3. **Tritium Beta Decay**

The tritium nucleus is radioactive and decays to He^3 with the emission of an electron and an antineutrino ($T \rightarrow He^3 + e^- + \bar{\nu}$).

(a) Assuming that the electron in the tritium atom is originally in its ground state, what is the probability of finding the electron in the resulting He^3 ion also in its $1s$ ground state immediately after the decay? You can assume that, as far as the electron is concerned, all that happens is that the nucleus suddenly changes its charge from $+1$ to $+2$. The other newly-produced electron is emitted with such high energy that it effectively leaves the atom immediately.

(b) What are the probabilities of finding the electron in the He^3 ion in each of its three $2p$ excited states immediately after the decay?

(c) What is the expectation value of the energy of the atomic electron immediately after the decay (old wave function, new Hamiltonian)?

Section B. Statistical Mechanics and Thermodynamics

1. **Putting Pressure on** ^3He

   Usually it is true that the entropy S of a solid is lower than that of the corresponding liquid. He^3 represents a counter example. Above 0.1 K to a good approximation *liquid* He^3 may be treated as a Fermi gas so S is proportional to the temperature. *Solid* He^3 which is stable at higher pressure, may be regarded as a regular lattice of non-interacting nuclear spins, with a constant nonzero spin entropy down to very low temperatures. The nuclei have spin 1/2.

   DATA:

$$
\begin{aligned}
S_{\text{liq}} &= S_{\text{solid}} \text{ at } 0.32 \text{ K}, \\
P_{\text{melt}} &= 31.0 \text{ atm at } 0.32 \text{ K}, \\
P_{\text{melt}} &= 33.0 \text{ atm at } 0.72 \text{ K}.
\end{aligned}
$$

   The volume change on melting is temperature independent at the low temperatures considered here.

   (a) Give an expression for the constant entropy of N atoms of *solid* He^3 in terms of fundamental constants.

   (b) Sketch the phase boundary between liquid and solid He^3 in the $P-T$ plane.

   (c) Evaluate P_{melt} at $T = 0$ K, assuming the above-described approximations remain valid there. Give P_{melt} to the nearest 0.1 atm. Explain how you obtain this result.

2. **Surface Adsorption**

Consider a 3-dimensional gas of spinless, non-relativistic, non-interacting bosons of mass m at pressure P and temperature T. The pressure of this 3D ideal Bose gas is low enough so that it is in the classical limit where the quantum statistics of the bosons may be neglected. The bosons can be adsorbed onto a 2-dimensional surface layer, where they are bound with energy $-\epsilon_0 < 0$, but retain their translational degrees of freedom in 2 dimensions. The ideal 3D Bose gas is in equilibrium with the ideal 2D adsorbed Bose gas. Treating the 2D adsorbed gas fully quantum mechanically with the proper Bose statistics, compute the surface density of this 2D gas as a function of the given parameters and fundamental constants.

(You may need: $\int \frac{dx}{ae^x+1} = \ln \frac{e^x}{1+ae^x}$.)

3. **Photon Condensate**

Consider a cavity that has two low-lying electromagnetic modes, with photon energies ϵ_0 and ϵ_1 (and energy difference $\Delta = \epsilon_1 - \epsilon_0 > 0$), that are separated by a large energy gap from all higher modes such that we can neglect higher mode contributions to any thermodynamic quantity. The quantum state of this cavity is specified by giving the numbers N_0 and N_1 of photons in the two modes.

First, suppose that these two modes exchange energy with a heat bath at temperature T in such a way that the total number $N = N_0 + N_1$ of photons in these two cavity modes (but not their individual numbers) is conserved. This is an idealization of an actual experimental setup involving laser optics.

(a) There are $N+1$ possible states of the cavity, labeled by the number N_1 of photons in the upper state ($N_1 = 0, 1, \ldots, N$). According to Boltzmann statistics, what are the occupation probabilities $p(N_1)$ of these states at bath temperature T?

(b) Your result from (a) simplifies in the limit of large N. Calculate the expectation value of the number of photons in the upper state in this limit, showing that it remains finite when $\Delta > 0$. The remaining photons go into the lower energy mode, which thus becomes a kind of Bose-Einstein condensate in this large N limit.

Now suppose that instead the system can exchange not only energy with the heat bath, but also photons, so that the total photon number N is not fixed. We must now use the grand canonical ensemble, an approach that involves a chemical potential parameter μ that we adjust to achieve the desired mean photon number. Note that even though these are photons, allow $\mu \neq 0$ in order to fix $\langle N \rangle$.

(c) Show that to have a large mean photon number $\langle N \rangle$, we must set $\mu = \epsilon_0 - k_B T / \langle N \rangle + ...$, where the neglected terms (....) are small compared to those shown explicitly. In the limit of large $\langle N \rangle$, does this ensemble have a different limiting value of $\langle N_1 \rangle$ from what you obtained in this limit in part (b)?

(d) Finally, in the large photon number limit, show that the fluctuations in the total photon number (and therefore in the photon number of the lower mode) are enormous in this grand canonical ensemble: $\langle (\delta N)^2 \rangle \cong \langle N \rangle^2$.