Prelims Solutions

Problem M15T3

Valentin Skoutnev

1

Since all the degrees of freedom are independent, the partition function of one particle will just be the product of the classical partition function $Z_1 = n_Q V$ and that of the magnetic interaction $Z_m = e^{\beta\mu H} + e^{-\beta\mu H} = 2cosh(\beta\mu H)$ where have absorbed any factors into μ. The total partition function for N particles is thus $Z_N = (Z_1 Z_m)^N/N!$ where we divide by $N!$ to account for the indistinguishablility of the particles.

2

$$U = -\frac{\partial ln(Z)}{\partial\beta} = \frac{3}{2}Nk_BT - N\mu H tanh(\frac{\mu H}{k_BT})$$

$$S = \beta(U-F) = \beta U + ln(Z) = \frac{3}{2}N - N\frac{\mu H}{k_BT}tanh(\frac{\mu H}{k_BT}) + Nln(n_QV) + Nln(2cosh(\frac{\mu H}{k_BT})) - Nln(N)$$

3

Slow (reversible) adiabatic process means the entropy stays constant. $dS = 0 = \frac{\partial S}{\partial T}dT + \frac{\partial S}{\partial H}dH \rightarrow \frac{\partial T}{\partial H} = -\frac{\frac{\partial S}{\partial H}}{\frac{\partial S}{\partial T}} = (k_BT)^2\frac{\frac{\partial S}{\partial H}}{\frac{\partial S}{\partial \beta}}$. Computing gives:

$$(k_BT)^2\frac{\frac{\partial S}{\partial H}}{\frac{\partial S}{\partial \beta}} = (k_BT)^2\frac{-N\frac{\mu}{k_BT}tanh(\frac{\mu H}{k_BT}) - N\frac{\mu^2 H}{(k_BT)^2}sech(\frac{\mu H}{k_BT})^2 + N\frac{\mu}{k_BT}tanh(\frac{\mu H}{k_BT}))}{-N\frac{\mu}{k_BT}tanh(\frac{\mu H}{k_BT}) - N\frac{\mu^2 H}{(k_BT)^2}sech(\frac{\mu H}{k_BT})^2 - \frac{3}{2}Nk_BT + N\frac{\mu}{k_BT}tanh(\frac{\mu H}{k_BT}))}$$

$$= (k_BT)^2\frac{N\frac{\mu^2 H}{(k_BT)^2}sech(\frac{\mu H}{k_BT})^2}{N\frac{\mu^2 H}{(k_BT)^2}sech(\frac{\mu H}{k_BT})^2 + \frac{3}{2}Nk_BT} > 0$$

Clearly $\frac{\partial T}{\partial H}$ is continuous and positive for all T, H. So as H continuously decreases so does T.

4

$S(T_i) = S(T_f) \rightarrow lnT_i^{\frac{3}{2}} - \frac{\mu H}{k_BT_i}tanh(\frac{\mu H}{k_BT_i}) + ln(cosh(\frac{\mu H}{k_BT_i})) = lnT_f^{\frac{3}{2}}$. We need to bound the terms that contain H. For H going to 0: $-\frac{\mu H}{k_BT_i}tanh(\frac{\mu H}{k_BT_i}) + ln(cosh(\frac{\mu H}{k_BT_i})) \rightarrow 0$. For H going to infinity: $-\frac{\mu H}{k_BT_i}tanh(\frac{\mu H}{k_BT_i}) + ln(cosh(\frac{\mu H}{k_BT_i})) \rightarrow -ln(2)$. Hence,

$$-ln(2) < -\frac{\mu H}{k_BT_i}tanh(\frac{\mu H}{k_BT_i}) + ln(cosh(\frac{\mu H}{k_BT_i})) < 0$$

which gives

$$S(T_f) = lnT_f^{\frac{3}{2}} = S(T_i) < lnT_i^{\frac{3}{2}} \rightarrow T_f < T_i$$

and

$$lnT_i^{\frac{3}{2}} - ln(2) < S(T_i) = S(T_f) = lnT_f^{\frac{3}{2}} \rightarrow 2^{-\frac{2}{3}}T_i < T_f$$