M15Q.3 Yukawa Potential - Born Approximation

Problem

Consider the scattering of a spinless particle of charge e and mass m in the screened Coulomb potential (also known as the Yukawa potential),

$$V(r) = -\frac{Ze^2}{r}e^{-r/a}$$

Compute the differential scattering cross section, $d\sigma/d\Omega$, for fast incident particles in the appropriate approximation. What is the range of validity for your approximation?

Solution

We are dealing with a high energy particle scattered from a relatively weak potential as indicated by the keywords "fast-moving" and "screened Coulomb potential". The first Born approximation is valid for large energies and weak scattering potentials, so we know the problem is urging us to use this.

The scattering cross section, $d\sigma/d\Omega$, is computed from the scattering amplitude, $f(\theta, \phi)$, like so:

$$\boxed{d\sigma/d\Omega = |f(\theta,\phi)|^2 = \frac{4\mu^2}{\hbar^4 q^2}\left|\int_0^\infty (r'V(r')sin(qr')dr')\right|^2}$$

In our case, we are going to assume the source of the potential is infinitely massive, so $\mu \to m$

$$\frac{4m^2}{\hbar^4 q^2}\left|\int_0^\infty (r'(-\frac{Ze^2}{r'}e^{-r'/a})sin(qr')dr')\right|^2 =$$
$$\frac{4m^2}{\hbar^4 q^2}\left|-Ze^2\int_0^\infty (e^{-r'/a})sin(qr')dr')\right|^2$$

Now just turn the crank

$$-Ze^2\int_0^\infty(e^{-r'/a})sin(qr')dr') = -Ze^2\int_0^\infty(e^{-r'/a})\frac{1}{2i}(e^{qr'}-e^{-qr'})dr')$$

$$= \frac{-Ze^2}{2i}\int_0^\infty(e^{r'(iq-1/a)}-e^{-r'(iq+1/a)})dr')$$

$$= \frac{-Ze^2}{2i}\left[\frac{e^{r(iq-1/a)}}{iq-1/a}+\frac{e^{-r(iq+1/a)}}{iq+1/a}\right]\Big|_0^\infty$$

Notice that both exponentials in the result have an oscillatory part and a decaying part, so we know that as $r\to\infty$ they will both converge to 0.

$$\frac{-Ze^2}{2i}\left[\frac{e^{r(iq-1/a)}}{iq-1/a}+\frac{e^{-r(iq+1/a)}}{iq+1/a}\right]\Big|_0^\infty = \frac{Ze^2}{2i}\left[\frac{1}{iq-1/a}+\frac{1}{iq+1/a}\right]$$

$$= \frac{Ze^2}{2i}\frac{-2iq}{q^2+1/a^2} = -Ze^2\frac{q}{q^2+1/a^2}$$

Now plugging back into the equation for the scattering amplitude, $f(\theta,\phi)$

$$\boxed{|f(\theta,\phi)|^2 = \frac{4m^2}{\hbar^4q^2}\left(Z^2e^4\frac{q^2}{(q^2+1/a^2)^2}\right) = \frac{4Z^2e^4m^2}{\hbar^4}\left(\frac{1}{(q^2+1/a^2)^2}\right)}$$

In order to get this in terms of θ, we need to recall where q comes from. q is the magnitude of the difference between the incoming wave vector, $\vec{k_0}$, and the outgoing wave vector, $\vec{k}$. In an elastic scattering process, which we'll assume this is, the magnitudes of $\vec{k_0}$ and $\vec{k}$ are the same. Therefore, we have the relation:

$$q = |\vec{k_0}-\vec{k}| = \sqrt{k_0^2+k^2-2kk_0cos(\theta)} = k\sqrt{2(1-cos(\theta)}$$

$$\text{note: } 1-cos(\theta) = 2sin^2(\theta)$$

$$\boxed{q = 2ksin(\theta/2)}$$

Note that $f(\theta,\phi)$is only a function of θ, as we have a spherical symmetric potential and thus a cylindrically symmetric scattering problem with respect to the incident particle's trajectory.
Now just plug in our relations for $d\sigma/d\Omega$.

$$d\sigma/d\Omega = \frac{4m^2Z^2e^4}{\hbar^4}\frac{1}{(4k^2sin^2(\theta/2)+1/a^2)^2}$$

What is the range of the validity of this approximation? Recall that in the first Born approximation we have plugged the zero-order solution, $\psi(\vec{r}) = \phi_{inc}(\vec{r})$ back into the integral form of the Schrödinger equation to get

$$\psi(\vec{r}) \approx \phi_{inc}(\vec{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|}V(\vec{r'})\phi_{inc}(\vec{r'})d^3r'$$

In order for this approximation to be valid, the second term has to be much smaller than $|\phi_{inc}(\vec{r})|^2$

$$\left|\frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|}V(\vec{r'})\phi_{inc}(\vec{r'})d^3r'\right|^2 \ll |\phi_{inc}(\vec{r})|^2$$

Since $\phi_{inc}(\vec{r'}) = e^{i\vec{k_0}\cdot\vec{r}}$ (incident plane wave) we can simplify the above equation a little further:

$$\left|\frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|}V(\vec{r'})e^{i\vec{k_0}\cdot\vec{r}}d^3r'\right| \ll 1$$

Again, assuming elastic scatter, we have $k_0 = k$. In order to simplify the integral a little further we will assume the scattering potential is largest near $r = 0$ so that

$$e^{ik|\vec{r}-\vec{r'}|} \to e^{ikr'} \text{ and } \frac{r'^2}{|\vec{r}-\vec{r'}|} \to r'$$

Now evaluate

$$\left|\frac{\mu}{\hbar^2}\int_0^\infty r'e^{ikr'}V(r')dr'\int_0^\pi e^{ikr'cos(\theta')}sin(\theta')d\theta'\right| \ll 1$$

The angular integral

$$\int_0^\pi e^{ikr'cos(\theta')}sin(\theta')d\theta' = \frac{1}{ikr'}e^{ikr'cos\theta}\Big|_0^\pi = \frac{1}{ikr'}\left(e^{-ikr'} - e^{ikr'}\right)$$

And now the final result for verifying the Born approximation:

$$\boxed{\frac{\mu}{\hbar^2 k}\left|\int\limits_0^\infty V(r')\left(e^{2ikr'}-1\right)dr'\right| \ll 1}$$

This is what we need to use in order to verify that our approximation is valid. Recall that we are using $\mu \to m$

$$I = \frac{mZe^2}{\hbar^2 k}\left|\int\limits_0^\infty \left(\frac{e^{-r/a}}{r}\right)\left(e^{2ikr}-1\right)dr\right|$$

If you write this integral down on the test, you should get a decent portion of the credit. Evaluating this integral should be something you come back to after doing the other problems.

Begin by evaluating $\dfrac{\partial I}{\partial(1/a)}$

$$\frac{\partial I}{\partial(1/a)} = \frac{mZe^2}{\hbar^2 k}\left|\left(-ae^{-r/a} - \frac{e^{r(2ik-1/a)}}{2ik-1/a}\right)\Big|_0^\infty\right|$$

$$\frac{\partial I}{\partial(1/a)} = \frac{mZe^2}{\hbar^2 k}\left|a + \frac{1}{2ik-1/a}\right|$$

Now integrate over $1/a$ knowing that as $1/a \to \infty$ we have $I \to 0$

$$I = \frac{mZe^2}{\hbar^2 k}\left|ln(1/a) - ln(1/a-2ik)\right| = \frac{mZe^2}{\hbar^2 k}\left|-ln(1-2ika)\right|$$

We need to do a little bit of fancy footwork to get the i out of the argument and into the coefficient so that when we take the magnitude it goes away.

$$-ln(1-2ika) = -ln\left(\frac{1-i(2ka)}{1+i(2ka)}(1+i(2ka))\right) =$$
$$-\frac{1}{2}ln\left(\frac{(1-i(2ka))^2}{(1+i(2ka))^2}(1+i(2ka)^2)\right)$$

I promise this is going somewhere.

$$-\frac{1}{2}ln\left(\frac{(1-i(2ka))^2}{(1+i(2ka))^2}(1+i(2ka)^2)\right) = -\frac{1}{2}ln\left(\frac{(1-i(2ka))}{(1+i(2ka))}(1+4k^2a^2))\right)$$

$$= -\frac{1}{2}ln\,(1+4k^2a^2) + itan^{-1}(2ka)$$

With this, we can write the validity condition as

$$\frac{mZe^2}{\hbar^2 k}\left| -\frac{1}{2}ln\,(1+4k^2a^2) + itan^{-1}(2ka)\right| =$$

$$\boxed{\frac{mZe^2}{\hbar^2 k}\left(\frac{1}{4}ln(1+4k^2a^2) + (tan^{-1}(2ka))^2\right)^{1/2} \ll 1}$$

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