# Problem M15M.3

(a) The centripetal force must be due entirely to friction, so that

$$f = MR\Omega^2,$$

where

$$\Omega = \frac{\ell}{R}\omega$$

is the angular velocity of the unicycle about the center of the turn. The normal force $N$ must be equal to the gravitational force $Mg$. Thus the torque on the wheel about its center is

$$\tau = \ell\,(2Mg\sin\phi - f\cos\phi),$$

which must be responsible for the precession of the angular momentum $\omega$ of the wheel. The vector $\omega$ changes at a rate $\Omega\,(\omega\cos\phi)$, which must be equal to the angular acceleration $\tau/I$. Combining all the above, we have

$$\ell M\left(2g\sin\phi - \frac{\ell^2\omega^2}{R}\cos\phi\right) = \tau = \frac{m\ell^3\omega^2}{R}\cos\phi,$$

which rearranges to

$$2MRg\sin\phi = \ell^2\omega^2\,(m+M)\cos\phi.$$

Thus we have

$$\phi = \boxed{\tan^{-1}\left(\frac{R\Omega}{2g}\frac{m+M}{M}\right)}.$$

In particular, the lean angle is independent of the size of the wheel.

(b) Assume the cyclist travels around a circle of radius 1 meter with a period of 2 seconds. Take a wheel of mass 1 kg, with a rider of mass 70 kg. Then $\phi \sim 9^\circ$, which is reasonable.

The wheel provides gyroscopic stability for the rider. If the lean angle is too large, then the wheel will precess faster due to increased graviational torque. This will decrease the turn radius, increasing the centrifugal force on the rider. This effect tends to straighten the unicycle and correct the error.

*Time: 25 m 3 s*