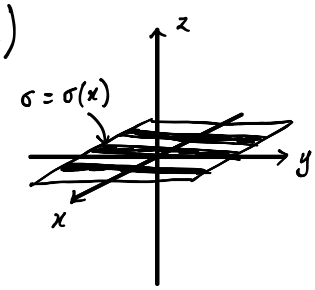
M15E.2 (Periodic Surface Charge Density)

(a)



ē boundary condit^{n} is $\sigma(x) = \sigma_0 \cos(kx)$

ē most straightforward route is to simply evaluate

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3\vec{r}\,' \rho(\vec{r}\,') \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

Here, $\rho(\vec{r}) = \sigma_0 \cos(kx)\, \delta(z)$, & $\vec{r} - \vec{r}\,' = \langle x - x', y - y', z - z' \rangle$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3\vec{r}\,' \sigma_0 \cos(kx') \frac{\langle x - x', y - y', z - z' \rangle}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} \delta(z')$$

By ē symmetry of ē problem, $\vec{E}$ cannot point in $\hat{y}$, so we can safely say $E_y = 0$. (also $z' = 0$).

$$E_z(\vec{r}) = \frac{\sigma_0}{4\pi\varepsilon_0} \int dx'\,dy' \cos(kx') \frac{z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}$$

ē integral over dy' can be written as: $\displaystyle\int_{-\infty}^{\infty} dy' \frac{\beta}{[(y-y')^2 + \alpha^2]^{3/2}}$ (let $y - y' = \alpha \tan\theta$, $dy' = -\alpha \sec^2\theta\, d\theta$

$$= \int_{-\pi/2}^{\pi/2} d\theta \frac{\alpha \sec^2\theta\, \beta}{\alpha^3 \sec^3\theta}$$

$$= \frac{\beta}{\alpha^2} \int_{-\pi/2}^{\pi/2} d\theta \cos\theta$$

$$= \frac{2\beta}{\alpha^2}$$

$$\Longrightarrow E_z(\vec{r}) = \frac{\sigma_0}{4\pi\varepsilon_0} \int dx' \cdot \frac{2z\cos(kx')}{(x-x')^2 + z^2}$$

$$= \frac{\sigma_0}{2\pi\varepsilon_0} z \int dx' \frac{\cos(kx')}{(x-x')^2 + z^2}$$

Equivalently, we can take x to be represented by some arbitrary phase on ē charge distribut^{n} instead.

$$\Longrightarrow E_z(\vec{r}) = \frac{\sigma_0}{2\pi\varepsilon_0} z \int dx' \frac{\cos(kx' + \varphi)}{x'^2 + z^2} \qquad (\varphi \equiv -kx)$$

$$= \frac{\sigma_0}{2\pi\varepsilon_0} z \cdot \frac{1}{z} e^{-kz} \pi \cos(\varphi) \qquad \text{(Mathematica, no idea how to compute this manually)}$$

$$= \frac{\sigma_0}{2\varepsilon_0} e^{-kz} \cos(kx).$$

ē x-component is similarly evaluated:

$$E_x(\vec{r}) = \frac{\sigma_0}{4\pi\varepsilon_0} \int dx'\,dy' \cos(kx') \frac{x - x'}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}$$

$$= \frac{\sigma_0}{2\pi\varepsilon_0} z \int dx' \frac{\cos(kx')(x-x')}{(x-x')^2 + z^2}$$

$$\equiv \frac{\sigma_0}{2\pi\varepsilon_0} z \int dx' \frac{x'\cos(kx' + \varphi)}{x'^2 + z^2}$$

$$= \frac{\sigma_0}{2\varepsilon_0} e^{-kz} \sin(kx)$$

$$\Longrightarrow \vec{E}(\vec{r}) = \frac{\sigma_0}{2\varepsilon_0} e^{-kz} \left(\sin(kx)\,\hat{x} + \cos(kx)\,\hat{z} \right)$$

(b) Now, w/ $\bar{c}$ object at $\vec{r} = \langle 0, 0, z_0 \rangle$, $\bar{c}$ electrostatic energy is:

$$U(z) = -\frac{1}{2}\alpha \vec{E}^2 = -\frac{\sigma_0^2 \alpha}{8\varepsilon_0^2} e^{-2kz}$$

$\bar{c}$ force is thus in $\bar{c}$ $\hat{z}$-direct$^{\underline{n}}$ as: $\vec{F} = -\frac{\partial U}{\partial z}$

$$= -\frac{\sigma_0^2 \alpha k}{4\varepsilon_0^2} e^{-2kz} \hat{z}$$