Problem M15E.3

For convenience, we complexify the drive field $\mathbf{B} = B_0 e^{i\omega t}\,\hat{\mathbf{z}}$.

The current per unit length $\mathbf{K}_{\text{eff}}$ around the cylinder satisfies

$$\mathbf{K}_{\text{eff}} = t\,\mathbf{J} = t\sigma\mathbf{E},$$

where $\mathbf{E} = E\,\hat{\boldsymbol{\phi}}$ is the electric field along the cylinder. This current generates a uniform solenoid field

$$\mathbf{B}_{\text{ind}} = \mu_0 K_{\text{eff}}\,\hat{\mathbf{z}}$$

within the cylinder, so the total field inside the cylinder is

$$\mathbf{B} = [B_0 e^{i\omega t} + \mu_0 t\sigma E(t)]\,\hat{\mathbf{z}}.$$

In steady-state we must have $E(t) = E_0 e^{i\omega t}$ for some complex E_0.

Maxwell's equations yield

$$2\pi a E_0 e^{i\omega t} = \oint_{\rho=a} \mathbf{E}\cdot \mathrm{d}\ell = -\iint_{\rho<a} \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\cdot \mathrm{d}A = -\pi a^2 [B_0 + \mu_0 t\sigma E_0]\, i\omega e^{i\omega t},$$

which simplifies to

$$2E_0 = -ai\omega[B_0 + \mu_0 t\sigma E_0].$$

This has the solution

$$E_0 = -\frac{ai\omega}{2 + ai\omega\mu_0 t\sigma} B_0,$$

and so the magnetic field inside the cylinder oscillates with amplitude

$$B' = |B_0 + \mu_0 t\sigma E_0| = B_0\left|\frac{2}{2 + ai\omega\mu_0 t\sigma}\right| = \boxed{\frac{2B_0}{\sqrt{4 + (a\omega\mu_0 t\sigma)^2}}}.$$

As expected, for $\omega \to 0$ we have $B' \to B_0$, and for $\omega \to \infty$ we have $B' \to 0$ (skin effect).