Problem M15E.1

(a) To leading-order, the power radiated per unit solid angle goes as

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{\mu_0}{16\pi^2 c}|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}|^2,$$

where $\mathbf{p} = -e\mathbf{x}$ is the dipole moment of the charge distribution and $\hat{\mathbf{r}} = \hat{\mathbf{x}}$ is the unit vector in the direction of propagation.

We have $\ddot{\mathbf{p}} = 0$ along the linear portions of the trajectory. During the semicircular segment, we have $|\ddot{\mathbf{p}}| = ea\omega_0^2$ pointing away from the origin. In the case $\hat{\mathbf{r}} = \hat{\mathbf{x}}$, we compute

$$\hat{\mathbf{r}} \times \ddot{\mathbf{p}} = -ea\omega_0^2 \cos(\omega_0 t)\,\hat{\mathbf{z}}.$$

Ignoring relativistic retardation, we thus obtain

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \boxed{\frac{\mu_0}{16\pi^2 c} e^2 a^2 \omega_0^4 \cos^2(\omega_0 t)}$$

for $0 < \omega_0 t < \pi$, and $\frac{\mathrm{d}P}{\mathrm{d}\Omega} = 0$ otherwise.

(b) During the semicircular segment, in this case

$$|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}| = ea\omega_0^2$$

since $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ is perpendicular to $\ddot{\mathbf{p}}$. Thus

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \boxed{\frac{\mu_0}{16\pi^2 c} e^2 a^2 \omega_0^4}$$

for $0 < \omega_0 t < \pi$, and $\frac{\mathrm{d}P}{\mathrm{d}\Omega} = 0$ otherwise.

(c) Again ignoring relativistic effects, the instantaneous radiated power is

$$P(t) = \frac{\mu_0}{16\pi^2 c}|\ddot{\mathbf{p}}|^2 \oiint_{S^2} \sin^2\theta \,\mathrm{d}\Omega\,,$$

where θ is the (polar) angle bewteen $\hat{\mathbf{r}}$ and $\ddot{\mathbf{p}}$. The spherical integral evaluates to

$$\oiint_{S^2} \sin^2\theta \,\mathrm{d}\Omega = \oiint_{S^2} (x^2 + y^2)\,\mathrm{d}\Omega = \frac{2}{3}\oiint_{S^2} (x^2 + y^2 + z^2)\,\mathrm{d}\Omega = \frac{8\pi}{3},$$

where we have symmetrized over $(x, y, z) \mapsto (y, z, x)$. Hence we recover the Larmor formula

$$P(t) = \frac{\mu_0}{6\pi c}|\ddot{\mathbf{p}}|^2 = \frac{\mu_0}{6\pi c} e^2 a^2 \omega_0^4$$

for $0 < \omega_0 t < \pi$. Thus the total energy radiated is

$$E = \int_0^{\pi/\omega_0} P(t)\,\mathrm{d}t = \boxed{\frac{\mu_0}{6c} e^2 a^2 \omega_0^3.}$$