We consider four spin-$S$ particles on a square that interact with nearest neighbors antiferromagnetically ($J > 0$).

\[ H = J(S_1 \cdot S_2 + S_2 \cdot S_3 + S_3 \cdot S_4 + S_4 \cdot S_1) \]

We are asked (a) to find a set of quantum numbers that classify eigenstates of $H$, (b) to compute energy levels and degeneracies for simplest case $S = 1/2$, (c) to compute energy, degeneracy, and quantum number for ground state of general spin $S$.

(a) The key insight is to notice

\[ H = J(S_{13} \cdot S_{24}) \]

by defining pairwise angular momenta

\[ S_{ij} = S_i + S_j \]

From here one can write

\[ H = \frac{J}{2}(S_{\text{total}}^2 - S_{13}^2 - S_{24}^2) \]

The quantum numbers are $S_T, S_{13}, S_{24}$. They uniquely define energy

\[ E = \frac{J}{2} [S_T(S_T + 1) - S_{13}(S_{13} + 1) - S_{24}(S_{24} + 1)] \]

(b) For $S = 1/2$, the pairs of spins have two choices $S_{13}, S_{24} \in 0, 1$, and the total spin has up to 3 choices $S_T \in 0, 1, 2$, subject to angular momentum addition rule

\[ S_i + S_j \in \{S_i + S_j, S_i + S_j - 1, \ldots, |S_i - S_j|\} \]

The degeneracy is defined by total angular momentum alone

\[ d = 2S_T + 1 \]

Tallying up the results

<table>
<thead>
<tr>
<th>$S_{13}$</th>
<th>$S_{24}$</th>
<th>$S_T$</th>
<th>$E[J]$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>-2</td>
<td>1</td>
</tr>
</tbody>
</table>

we find four distinct energy levels

\[ E_0 = -2J \quad d_0 = 1 \]
\[ E_1 = -J \quad d_1 = 3 \]
\[ E_2 = 0 \quad d_2 = 7 \]
\[ E_3 = J \quad d_3 = 5 \]

One way to check the answer is summing over degeneracies $\sum d_j = 2^4$. This is consistent with 4 independent particles that each have 2 possible states.
(c) For general spin $S$ we find the ground state by considering energy

$$E = \frac{J}{2} [S_T(S_T + 1) - S_{13}(S_{13} + 1) - S_{24}(S_{24} + 1)]$$

From inspection $E_0$ must minimize $S_T$ and maximize $S_{13}, S_{24}$. This is achieved when

$$S_{13} = S_{24} = 2S$$
$$S_T = 0$$

Therefore $E_0 = -J 2S(2S + 1)$ and $d_0 = 1$. \(^1\)

---

\(^1\)Note: This problem is identical to J99 Q2.
The key insight is to pick the right eigenbasis and find

$$H = J(S_{13} \cdot S_{24})$$

At first glance, we see four independent particles so the basis might be \{S_1, S_2, S_3, S_4\}. But this does not diagonalize dot product Hamiltonians with $S_i \cdot S_j$. At second glance, one might try a basis \{S_{12}, S_{23}, S_{34}, S_{41}\} and write

$$H = \frac{J}{2} \left[ (S_{12}^2 + S_{23}^2 + S_{34}^2 + S_{41}^2) - 2(S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_{13}^2 + S_{24}^2) \right]$$

This fails because the four pairwise angular momenta are over-constrained. For example it is impossible to have $(0, 0, 0, 1)$ or $(1, 1, 1, 0)$. Just try to draw it. We really need a basis where the spins are decoupled, this happens when each angular momentum partial sum appears only once. While it is mathematically trivial to write down

$$S_{13} \cdot S_{24} = (S_1 + S_3) \cdot (S_2 + S_4) = S_1 \cdot S_2 + S_2 \cdot S_3 + S_3 \cdot S_4 + S_4 \cdot S_1$$

factoring the sum in this way gives us a way to write down the Hamiltonian where each partial sum appears only once. The problem of 4 spins on a square is now transformed to 2 bosons on a line.