M13 Q3 : Spin on a Square **Tony M. Qian : Jan 2020**

We consider four spin-S particles on a square that interact with nearest neighbors antiferromagnetically ($J > 0$).

$$H = J(S_1 \cdot S_2 + S_2 \cdot S_3 + S_3 \cdot S_4 + S_4 \cdot S_1)$$

We are asked (a) to find a set of quantum numbers that classify eigenstates of H, (b) to compute energy levels and degeneracies for simplest case $S = 1/2$, (c) to compute energy, degeneracy, and quantum number for ground state of general spin S.

(a) The key insight is to notice

$$H = J(S_{13} \cdot S_{24})$$

by defining pairwise angular momenta

$$S_{ij} = S_i + S_j$$

From here one can write

$$H = \frac{J}{2}(S_{total}^2 - S_{13}^2 - S_{24}^2)$$

The quantum numbers are S_T, S_{13}, S_{24}. They uniquely define energy

$$E = \frac{J}{2}\left[S_T(S_T + 1) - S_{13}(S_{13} + 1) - S_{24}(S_{24} + 1)\right]$$

(b) For $S = 1/2$, the pairs of spins have two choices $S_{13}, S_{24} \in 0, 1$, and the total spin has up to 3 choices $S_T \in 0, 1, 2$, subject to angular momentum addition rule

$$S_i + S_j \in \{S_i + S_j, S_i + S_j - 1, \dots, |S_i - S_j|\}$$

The degeneracy is defined by total angular momentum alone

$$d = 2S_T + 1$$

Tallying up the results

S_{13}	S_{24}	S_T	$E[J]$	d
0	0	0	0	1
0	1	1	0	3
1	0	1	0	3
1	1	2	1	5
1	1	1	-1	3
1	1	0	-2	1

we find four distinct energy levels

$$\begin{aligned} E_0 &= -2J & d_0 &= 1 \\ E_1 &= -J & d_1 &= 3 \\ E_2 &= 0 & d_2 &= 7 \\ E_3 &= J & d_3 &= 5 \end{aligned}$$

One way to check the answer is summing over degeneracies $\sum d_j = 2^4$. This is consistent with 4 independent particles that each have 2 possible states.

(c) For general spin S we find the ground state by considering energy

$$E = \frac{J}{2}\left[S_T(S_T+1) - S_{13}(S_{13}+1) - S_{24}(S_{24}+1)\right]$$

From inspection E_0 must minimize S_T and maximize S_{13}, S_{24}. This is achieved when

$$S_{13} = S_{24} = 2S$$

$$S_T = 0$$

Therefore $E_0 = -J\,2S(2S+1)$ and $d_0 = 1$. [1]

[1] Note: This problem is identical to J99 Q2.
The key insight is to pick the right eigenbasis and find

$$H = J(S_{13} \cdot S_{24})$$

At first glance, we see four independent particles so the basis might be $\{S_1, S_2, S_3, S_4\}$. But this does not diagonalize dot product Hamiltonians with $S_i \cdot S_j$. At second glance, one might try a basis $\{S_{12}, S_{23}, S_{34}, S_{41}\}$ and write

$$H = \frac{J}{2}\left[(S_{12}^2 + S_{23}^2 + S_{34}^2 + S_{41}^2) - 2(S_1^2 + S_2^2 + S_3^2 + S_4^2 +)\right]$$

This fails because the four pairwise angular momenta are over-constrained. For example it is impossible to have $(0,0,0,1)$ or $(1,1,1,0)$. Just try to draw it. We really need a basis where the spins are decoupled, this happens when each angular momentum partial sum appears only once. While it is mathematically trivial to write down

$$S_{13} \cdot S_{24} = (S_1 + S_3) \cdot (S_2 + S_4) = S_1 \cdot S_2 + S_2 \cdot S_3 + S_3 \cdot S_4 + S_4 \cdot S_1$$

factoring the sum in this way gives us a way to write down the Hamiltonian where each partial sum appears only once. The problem of 4 spins on a square is now transformed to 2 bosons on a line.