Problem M13Q.2

(a) The Hamiltonian is

$$H = -\frac{\hbar^2}{2m}\partial_x^2 - V_0\,\delta(x),$$

so the Schrödinger equation reads

$$\Psi''(x) = \frac{2m\,(-E - V_0\,\delta(x))}{\hbar^2}\Psi(x). \tag{1}$$

To satisfy this equation for $x \neq 0$ with a normalizable wavefunction, we must have

$$\Psi(x) = Ae^{-|x|/x_0} \quad \text{where} \quad x_0 := \frac{\hbar}{\sqrt{-2mE}}. \tag{2}$$

Integrating equation (1) on a small interval around $x = 0$ yields

$$\Psi'(x)|_{x\to 0^+} - \Psi'(x)|_{x\to 0^-} = -\frac{2mV_0}{\hbar^2}\Psi(0). \tag{3}$$

Using the form (2) of the wavefunction, equation (3) becomes

$$\frac{2}{x_0} = \frac{2mV_0}{\hbar^2}, \quad \text{which implies} \quad E = \boxed{-\frac{mV_0^2}{2\hbar^2}.}$$

It remains to normalize the wavefunction. We impose the normalization

$$1 = \int_{-\infty}^{\infty} \Psi(x)^2\,\mathrm{d}x = 2A^2\int_0^{\infty} e^{-2x/x_0}\,\mathrm{d}x = A^2 x_0,$$

so that

$$\Psi(x) = \boxed{\frac{e^{-|x|/x_0}}{\sqrt{x_0}}.}$$

(b) The potential can only couple the ground state to continuum states with odd spatial parity. Thus we parameterize the continuum wavefunctions by

$$\Psi_k(x) = \frac{1}{\sqrt{\pi}}\sin(kx), \quad \text{with eigen-energies} \quad E_k = \frac{\hbar^2 k^2}{2m},$$

where we have enforced delta-function normalization. Since $\Psi_k(x) = 0$, the $-V_0\,\delta(x)$ term in the Hamiltonian does not contribute, and so the $|\Psi_k\rangle$ are indeed eigenfunctions of H.

By Fermi's golden rule, the transition rate is

$$P_{g\to k} = \frac{2\pi}{\hbar}|\langle k|A|g\rangle|^2\,\rho(E_k),$$

where k is chosen so that $E_k - E = \hbar\omega$, and

$$A := \frac{Fx}{2}$$

is the amplitude of the co-rotating term $e^{i\omega t}$ in V_1. We directly compute

$$\begin{aligned}\langle k|x|g\rangle &= \frac{2}{\sqrt{\pi x_0}}\int_0^{\infty} x\sin(kx)e^{-x/x_0}\,\mathrm{d}x \\ &= \frac{4kx_0^3}{(k^2x_0^2+1)^2\,\sqrt{\pi x_0}},\end{aligned}$$

It remains to compute the density of states. Placing the system in a box $|x| < L$, the values of k are quantized to satisfy that $kL = n\pi$ for some $n \in \mathbb{N}$. It follows that

$$\mathrm{d}E_k = \frac{\hbar^2 k}{m}\,\mathrm{d}k = \frac{\hbar^2 k}{m}\frac{\pi}{L}\,\mathrm{d}n\,, \quad \text{and so} \quad \rho(E_k) = \frac{1}{L}\frac{\mathrm{d}n}{\mathrm{d}E_k} = \frac{m}{\hbar^2 k\pi}.$$

Combining the above, we obtain

$$P_{g\to k} = \frac{F^2 m}{2\hbar^3 k}|\langle k|x|g\rangle|^2 = \boxed{\frac{8F^2 m k x_0^5}{\pi\hbar^3(k^2x_0^2+1)^4}},$$

which has units of Hz as expected.